Worksheet 3

Problem 1. Consider the matrix

$$A = \begin{pmatrix} 1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(1) Describe in words the linear transformation T_A.
(2) Compute the inverse A^{-1} of A.
(3) Describe in words the linear transformation $T_{A^{-1}}$.

Problem 2. (1) Is there an injective linear transformation $\mathbb{R}^3 \to \mathbb{R}^2$?
(2) Is there a surjective linear transformation $\mathbb{R}^2 \to \mathbb{R}^3$?

Problem 3. (1) Let A, B be two invertible square matrices of size n. Show that AB is invertible and that

$$(AB)^{-1} = B^{-1}A^{-1}.$$

(2) Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$. Compute AB, $(AB)^{-1}$, A^{-1}, B^{-1}, $A^{-1}B^{-1}$ and $B^{-1}A^{-1}$. Notice that

$$(AB)^{-1} = B^{-1}A^{-1} \neq A^{-1}B^{-1}.$$

Problem 4. (1) Let A be a square matrix of size n. Show that if there is a matrix B such that $BA = 0$ then A is not invertible.
(2) Let

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Compute J^5. Is J invertible ?

Problem 5. (1) Let M be a square matrix and I the identity matrix of the same size. For a a real number, compute $(I - aM)(I + aM + a^2M^2 + a^3M^3 + a^4M^4)$.
(2) Let

$$A = \begin{pmatrix} 1 & -a & 0 & 0 & 0 \\ 0 & 1 & -a & 0 & 0 \\ 0 & 0 & 1 & -a & 0 \\ 0 & 0 & 0 & 1 & -a \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

where a is a real number. Show that A is invertible and compute A^{-1}. *Hint : express A in terms of the identity matrix I and of J (of the previous problem).*

Problem 6. Let

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & -1 \end{pmatrix}$$

(1) Express A^3 in terms of A and the identity matrix I.
(2) Show that A is invertible and give its inverse.