Worksheet 2

Problem 1. Let $h \in \mathbb{R}$. and

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 1 & h \\ -4 & 4 & -4 \\ 6 & 4 & 0 \end{pmatrix}$$

. Let T_A the linear map attached to A.

(1) What is the source space of T_A ? The target space ?

(2) Which of the following vectors is in the range of T_A ?

$$\text{a)} \begin{pmatrix} -4 \\ 4 \\ -4 \end{pmatrix} \text{ b)} \begin{pmatrix} 4 \\ 2 \\ 1+h \end{pmatrix} \text{ c)} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ d)} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \text{ e)} \begin{pmatrix} 3 \\ 1 \\ -4 \\ 6 \end{pmatrix} \text{ f)} \begin{pmatrix} 4 \\ 2 \\ 0 \\ 10 \end{pmatrix} \text{ g)} \begin{pmatrix} 3 \\ 2 \\ 4 \\ -2 \end{pmatrix} \text{ h)} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

(3) Is T_A surjective ?

(4) Suppose that $h = 0$. Is T_A injective ?

(5) (bonus question) Show that T_A is injective for any $h \in \mathbb{R}$.

Problem 2. True/False

(1) Let A be a 3 x 2 matrix. Suppose that $A\mathbf{x} = \mathbf{0}$ has a unique solution. Then $A\mathbf{x} = \mathbf{b}$ has a solution for any $\mathbf{b} \in \mathbb{R}^3$. False. The first sentence means that the corresponding linear map $T_A : \mathbb{R}^2 \to \mathbb{R}^3$ is injective. It does not imply that T_A is surjective (which is what the second statement means). Counter example : $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$

(2) If $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is an independent family of vectors then $\{\mathbf{u}, \mathbf{v}\}$ is an independent family of vectors.

True. We discussed it in class.

(3) If $\{\mathbf{u}, \mathbf{v}\}$ is independent, $\{\mathbf{u}, \mathbf{w}\}$ is independent, and $\{\mathbf{v}, \mathbf{w}\}$ is independent, then $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is independent. False. We discussed it in class. Counter example : $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

(4) The homogeneous system attached to the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ has infinitely many solutions. True.

(5) The family of vectors $\{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\}$ spans $\mathbb{R}^3$. True.

(6) The family of vectors $\{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}\}$ is independent but does not span $\mathbb{R}^3$.

False. It is independent and it does span $\mathbb{R}^3$ (take a vector of the form $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ and show that it is a linear combination of the 3 vectors.)

(7) The family of vectors $\{\begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}\}$ is independent but does not span $\mathbb{R}^3$. True.

For example, $\begin{pmatrix}0\\1\\0\end{pmatrix}$ is not in the Span of the 3 vectors.

(8) The homogeneous system attached to the matrix $A = \begin{pmatrix}1 & 0 & 1\\0 & 1 & 1\\1 & 1 & 3\end{pmatrix}$ has infinitely many solutions. False.

(9) The homogeneous system attached to the matrix $A = \begin{pmatrix}1 & 0 & 1\\0 & 1 & 1\\1 & 1 & 2\end{pmatrix}$ has infinitely many solutions. True.

(10) The homogeneous system attached to the matrix $A = \begin{pmatrix}1 & 0 & 1 & 0\\0 & 1 & 1 & 2\\1 & 1 & 3 & -1\end{pmatrix}$ has infinitely many solutions. True.

(11) The linear map attached to the matrix $A = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ is the identity $\mathbb{R}^2 \to \mathbb{R}^2$. True.

(12) The linear map attached to the matrix $A = \begin{pmatrix}-1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}$ is the orthogonal reflection about the $(0, y, z)$ plane. True.

(13) The linear map $\mathbb{R}^2 \to \mathbb{R}^2$ attached to the matrix $A = \begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}$ is the rotation by $\pi/2$. False, it is the rotation by $-\pi/2$.