1. TRUE or FALSE

   (a) A homogenous system with more variables than equations has a nonzero solution.
   True *(The number of pivots is going to be less than the number of columns and therefore there is a free variable which provides nonzero solutions.)*

   (b) If $\mathbf{x}$ is a solution to the system of equations $A\mathbf{x} = \mathbf{b}$ for some vector $\mathbf{b}$ then $2\mathbf{x}$ is also a solution to the same system.
   False *(In fact whenever $\mathbf{b}$ is nonzero and $\mathbf{x}$ is a solution to $A\mathbf{x} = \mathbf{b}$ then $A(2\mathbf{x}) = 2A\mathbf{x} = 2\mathbf{b} \neq \mathbf{b}$.)*

   (c) If A has a zero column then the homogenous system $A\mathbf{x} = \mathbf{0}$ has a nonzero solution.
   True *(The zero column will contain no pivot and therefore the corresponding variable is free and provides nonzero solutions.)*

   (d) The only linear transformation which is both one-to-one and onto is the identity map.
   False *(For example it is easy to see that the map $T : \mathbb{R}^2 \to \mathbb{R}^2$ which takes every vector $\mathbf{v} \in \mathbb{R}^2$ to $-\mathbf{v}$ is a linear transformation which is both one-to-one and onto.)*

2. Find the complete solution set to the system:

$$\begin{cases} 2x_1 + 2x_2 + x_3 + x_4 - x_5 = 6 \\ 2x_1 + 2x_2 + x_3 + 2x_4 - 3x_5 = 10 \\ -2x_1 - 2x_2 + 2x_3 - 3x_4 + 6x_5 = -12 \end{cases}$$

$$\left(\begin{array}{rrrrr|r} 2 & 2 & 1 & 1 & -1 & 6 \\ 2 & 2 & 1 & 2 & -3 & 10 \\ -2 & -2 & 2 & -3 & 6 & -12 \end{array}\right) \to \left(\begin{array}{rrrrr|r} 2 & 2 & 1 & 1 & -1 & 6 \\ 0 & 0 & 0 & 1 & -2 & 4 \\ 0 & 0 & 3 & -2 & 5 & -6 \end{array}\right) \to$$

$$\left(\begin{array}{rrrrr|r} 2 & 2 & 1 & 1 & -1 & 6 \\ 0 & 0 & 3 & -2 & 5 & -6 \\ 0 & 0 & 0 & 1 & -2 & 4 \end{array}\right)$$

Using the row echelon form of the matrix given above, we see that x_2, x_5 are free and we have:

$$\begin{cases} 2x_1 + 2x_2 + x_3 + x_4 - x_5 = 6 \\ 3x_3 - 2x_4 + 5x_5 = -6 \\ x_4 - 2x_5 = 4 \end{cases} \Rightarrow$$

$$\begin{cases} x_4 = 4 + 2x_5 \\ 3x_3 = -6 + 2x_4 - 5x_5 = -6 + 2(4 + 2x_5) - 5x_5 = 2 \cancel{+ 3x_5} - x_5 \\ 2x_1 = 6 - 2x_2 - x_3 - x_4 + x_5 = 6 - 2x_2 - (1/3)(2 \cancel{+ 3x_5} - x_5) - (4 + 2x_5) + x_5 = 4/3 - 2x_2 - \cancel{2x_5} \tfrac{2}{3}x_5 \end{cases}$$

So the general solution is of the form

$$\begin{cases} x_1 = 2/3 - x_2 - \cancel{x_5}\, x_5/3 \\ x_3 = 2/3 \cancel{+ x_5} - x_5/3 \\ x_4 = 4 + 2x_5 \end{cases}$$

for arbitrary x_2 and x_5.

3. Find the reduced row echelon matrix which is row equivalent to

$$\begin{pmatrix} 3 & 3 & 3 & 3 \\ 0 & 1 & 2 & 3 \\ 2 & 8 & 14 & 10 \\ 1 & 3 & 5 & 12 \end{pmatrix}$$

row reducing:

$$\begin{pmatrix} 3 & 3 & 3 & 3 \\ 0 & 1 & 2 & 3 \\ 2 & 8 & 14 & 10 \\ 1 & 3 & 5 & 12 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 6 & 12 & 8 \\ 0 & 2 & 4 & 11 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & -10 \\ 0 & 0 & 0 & 5 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

4. Suppose

$$A = \begin{pmatrix} 1 & -3 & 0 & 2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(a) Find all the solutions to the system of equations

$$A\mathbf{x} = \begin{pmatrix} 2 \\ -4 \\ 0 \end{pmatrix}$$

Solving for the system with augmented matrix

$$\left(\begin{array}{ccccc|c} 1 & -3 & 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & 3 & 0 & -4 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -3 & 0 & 2 & -1 & 2 \\ 0 & 0 & 1 & 3 & 0 & -4 \end{array}\right) \longrightarrow \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -2/3 & 1/3 & -2/3 \\ 0 & 0 & 1 & 3 & 0 & -4 \end{array}\right)$$

Free variables are x_4 and x_5

$x_1 = -x_5$ $\quad$ $x_2 = -\frac{2}{3} + \frac{2}{3}x_4 - \frac{1}{3}x_5$ $\quad$ $x_3 = -4 - 3x_4$

(b) Express the vector

$$\begin{pmatrix} 2 \\ -4 \\ 0 \end{pmatrix}$$

as a linear combination of columns of A. We use the solution found in part (a).
Set $x_4 = x_5 = 0$. Then $x_1 = 0$, $x_2 = -\frac{2}{3}$, $x_3 = -4$. Now we can check that.

$$0\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \frac{2}{3}\begin{pmatrix} -3 \\ 0 \\ 0 \end{pmatrix} - 4\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 0\begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} + 0\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ 0 \end{pmatrix}$$

5. In each of the following cases, determine whether the given vector is in the set spanned by the columns of the given matrix:

(a) $\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ with $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{pmatrix}$

We want to determine the consistency of the linear system whose augmented matrix is:

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & -1 & 1 \\ 1 & 1 & 0 & 3 \end{array}\right)$$

Row reducing: $\longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & 0 \end{array}\right)$

The system is consistent, and therefore the vector $\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ is in the span of the columns of the matrix.

(b) $\begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix}$ with $\begin{pmatrix} 3 & 2 & -4 \\ 1 & -1 & -3 \\ 1 & 5 & 3 \end{pmatrix}$ Same reasoning as in (a)

$$\left(\begin{array}{ccc|c} 3 & 2 & -4 & 4 \\ 1 & -1 & -3 & 0 \\ 1 & 5 & 3 & -3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & -1 & -3 & 0 \\ 0 & 6 & 6 & -3 \\ 0 & 5 & 5 & 4 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & -1 & -3 & 0 \\ 0 & 1 & 1 & -\frac{1}{2} \\ 0 & 1 & 1 & \frac{4}{5} \end{array}\right)$$

This system is inconsistent, and therefore $\begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix}$ is not a linear combination of the columns of the matrix.

6. Determine if the set $S = \left\{ \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \right\}$ is linearly independent in $\mathbb{R}^3$ and explain why.

The set S is linearly independent if and only if

$$c_1 \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + c_3 \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ implies that } c_1 = c_2 = c_3 = 0.$$

We need to check that the only solution to $\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ is the trivial one.

We can now reduce the matrix:

$$\longrightarrow \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

As the matrix has a non-pivot column, the system has nontrivial solutions, and therefore the set S is linearly dependent.

7. Determine if each of the following functions is a linear transformation. If it is the case find the matrix representing the transformation with respect to the standard bases.

(a) $L : \mathbb{R}^3 \to \mathbb{R}^1$, with $L\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = -x_2 - x_1$.

Solution: Suppose $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ are vectors in $\mathbb{R}^3$ and $c \in \mathbb{R}$ is a scalar. Then

$$\begin{aligned} L(c\,\mathbf{x}) &= L\left(\begin{pmatrix} c\,x_1 \\ c\,x_2 \\ c\,x_3 \end{pmatrix}\right) = -c\,x_2 - c\,x_1 = c\,(-x_2 - x_1) \\ &= c\,L(\mathbf{x}) \end{aligned}$$

Also

$$\begin{aligned} L(\mathbf{x}+\mathbf{y}) &= L\left(\begin{pmatrix} x_1+y_1 \\ x_2+y_2 \\ x_3+y_3 \end{pmatrix}\right) = -(x_2+y_2) - (x_1+y_1) = (-x_2-x_1) + (-y_2-y_1) \\ &= L(\mathbf{x}) + L(\mathbf{y}) \end{aligned}$$

These show that L is linear. Also we can see that $L(\mathbf{e_1}) = -1, L(\mathbf{e_2}) = -1$ and $L(\mathbf{e_3}) = 0$ for the standard vectors in $\mathbb{R}^3$, which shows that the standard matrix is the 1×3 matrix $\begin{pmatrix} -1 & -1 & 0 \end{pmatrix}$.

(b) $L : \mathbb{R}^2 \to \mathbb{R}^2$, with $L\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} x_1x_2 \\ x_2 \end{pmatrix}$.

Solution: L is not linear. For example we can see that $L(\mathbf{e}_1 + \mathbf{e}_2) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. But $L(\mathbf{e}_1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $L(\mathbf{e}_2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, so

$$L(\mathbf{e}_1 + L(\mathbf{e}_2) \neq L(\mathbf{e}_1) + L(\mathbf{e}_2)$$

which shows that L is not linear.

(c) $L : \mathbb{R}^3 \to \mathbb{R}$ with $L\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = 1$.

Solution: L is not linear. For example $L(2\mathbf{e}_1) = 1 \neq 2L(\mathbf{e}_1)$.

(d) $L : \mathbb{R}^2 \to \mathbb{R}^2$ with $L\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} x_2 - x_1 \\ 0 \end{pmatrix}$.

Solution: It is easy to check like part (a) above to see that L is linear. To find the standard matrix, we have

$$L(\mathbf{e_1}) = \begin{pmatrix} -1 \\ 0 \end{pmatrix} = (-1)\mathbf{e}_1 + 0\mathbf{e}_2 \quad \text{and} \quad L(\mathbf{e_2}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathbf{e}_1 + 0\mathbf{e}_2$$

So the standard matrix of L is $\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$.

8. For the following linear transformations, find the standard matrix and also determine if they are one-to-one or onto.

(a) $T : \mathbb{R}^3 \to \mathbb{R}$ with $T\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = x_2$.

Solution: It follows from the definition of T that $T(\mathbf{e}_1) = T(\mathbf{e}_3) = 0$ and $T(\mathbf{e}_2) = 1$. So the standard matrix of T is the 1×3 matrix $A_T = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}$.

T is onto because for every $b \in \mathbb{R}$, $T\left(\begin{pmatrix} 0 \\ b \\ 0 \end{pmatrix}\right) = b$. In other terms, $A_T\mathbf{x} = b$ is consistent for every $b \in \mathbb{R}$.

T is not one-to-one, since $T(\mathbf{e}_1) = T(\mathbf{0}) = 0$, or in other terms $A_T\mathbf{x} = 0$ has more than one solution.

(b) $T : \mathbb{R}^2 \to \mathbb{R}^3$ with $T(\mathbf{e}_1) = \mathbf{e}_2 + \mathbf{e}_3$ and $T(\mathbf{e}_2) = -\mathbf{e}_1 + \mathbf{e}_2$.

Solution: The definition of T immediately shows that the standard matrix of T is
$A_T = \begin{pmatrix} 0 & -1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}$

T is not onto because there is no vector $\mathbf{x} \in \mathbb{R}^2$ so that $T(\mathbf{x}) = \mathbf{e}_1$, or in other terms $A_T\mathbf{x} = \mathbf{e}_1$ is inconsistent.

T is one-to-one. Recall that to prove this one only needs to show that the homogenous system $A_T\mathbf{x} = \mathbf{0}$ has only the trivial solution. This can be seen by finding the row echelon form of the matrix has two pivots.

(c) $T : \mathbb{R}^2 \to \mathbb{R}^2$ with $T\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} x_1 + x_2 \\ -x_1 - x_2 \end{pmatrix}$.

Solution: $T(\mathbf{e}_1) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \mathbf{e}_1 - \mathbf{e}_2$ and $T(\mathbf{e}_2) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \mathbf{e}_1 - \mathbf{e}_2$. So the standard matrix of T is $A_T = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$.

T is not onto, because $T(\mathbf{x}) = \mathbf{e}_1$ has no solutions, or in other terms $A_T\mathbf{x} = \mathbf{e}_1$ is inconsistent.

T is not one-to-one either, because $T(\mathbf{e}_1 - \mathbf{e}_2) = T(\mathbf{0}) = \mathbf{0}$, in other terms $A_T\mathbf{x} = \mathbf{0}$ has more than one solutions.

9. Suppose the following vectors in $\mathbb{R}^3$ are given

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

(a) Determine if the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

Solution: Recall that to check that S is linearly independent, we need to check if the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$ has a nonzero solution. Equivalently we need to check if the homogenous system of equations $A\mathbf{x} = \mathbf{0}$ has a nonzero solution, where

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

To show this system has no nonzero solution, it is enough to find the reduced row echelon form of A and see that it has exactly three pivots.

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \to$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Hence S is linearly independent.

(b) Determine if S spans $\mathbb{R}^3$.

Solution: To prove S spans $\mathbb{R}^3$, we need to show the system of linear equations $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b} \in \mathbb{R}^3$. But we saw that A has three pivots and therefore every row contains a pivot. This implies that $A\mathbf{x} = \mathbf{b}$ is consistent.

(c) Express the vector

$$\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$$

as a linear combination of elements of the vectors in S.

Solution: We need to find a solution to the system of equations

$$A\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 3 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 3 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & -1 & 2 \end{array}\right) \to$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -2 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1/2 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3/2 \\ 0 & 1 & 0 & 3/2 \\ 0 & 0 & 1 & -1/2 \end{array}\right)$$

This shows

$$\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = \frac{3}{2}\mathbf{v}_1 + \frac{3}{2}\mathbf{v}_2 - \frac{1}{2}\mathbf{v}_3.$$

10. Suppose $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation which is not onto. Answer the following questions and explain your answers.

(a) What is the size of the standard matrix for T?

$n \times n$

(b) How many pivots does the standard matrix of T has?

The number of pivots of the standard matrix of T is strictly less than the number of its rows, n, as T is not onto.

(c) Can T be one-to-one?

Since the standard matrix has less than n pivots (as observed in (b)), it does not have a pivot in every column. Therefore T cannot be one-to-one.