Math 221, Term 2, 2017–2018

Instructor: R. Ollivier (Section 202)

Thursday, March 22, 2018

Instructions

- This is a closed book/notes exam. Use of calculators is not permitted.
- You have **50 minutes.**
- Please do all your work on the exam provided.
- You must show your work to receive full credit on a problem (except for the multiple choice questions).

Last name: ______________________________

First name: ______________________________

Student #: ______________________________

Signature: ______________________________

Answer Key

Grader's use only:

1. ______ /10

2. ______ /10

3. ______ /15

4. ______ /10

5. ______ /10

1. [10 points] Compute the determinant of the following matrix and determine if it is invertible. As usual, remember to show your work!

$$A = \begin{pmatrix} 3 & -2 & 1 \\ 6 & 0 & 2 \\ -1 & 3 & -2 \end{pmatrix}$$

$$\det A = 3\begin{vmatrix} 0 & 2 \\ 3 & -2 \end{vmatrix} + 2\begin{vmatrix} 6 & 2 \\ -1 & -2 \end{vmatrix} + 1\begin{vmatrix} 6 & 0 \\ -1 & 3 \end{vmatrix}$$

$$= 3 \times (-6) + 2(-12 + 2) + 1 \times 18$$

$$= -20 \implies A \text{ is invertible.}$$

Question 2 Continued:

$L_1 \leftarrow L_1 + 2L_2$.

$$\begin{bmatrix} 1 & -0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \underbrace{\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 2 \\ -3 & 4 & 5 \end{bmatrix}}_{A^{-1}}$$

Double check:

$$\begin{bmatrix} 3 & -1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 2 \\ -3 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

2. [10 points]

Find the inverse of the following matrix

$$A = \begin{pmatrix} 3 & -1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$\begin{bmatrix} 3 & -1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ $\quad L_1 \longleftrightarrow L_2. \quad$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & -2 & 1 \\ 3 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ $\quad \begin{array}{l} L_2 \leftarrow L_2 - 3L_1 \\ L_3 \leftarrow L_3 - L_1 \end{array} \quad$ $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 5 & -2 \\ 0 & 3 & -1 \end{bmatrix}$ $\quad L_3 \leftarrow L_3 - \frac{3}{5} L_2 \quad$ $\begin{bmatrix} 0 & 1 & 0 \\ 1 & -3 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 5 & -2 \\ 0 & 0 & -1+\frac{6}{5} \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 5 & -2 \\ 0 & 0 & \frac{1}{5} \end{bmatrix}$ $\quad$ $\begin{bmatrix} 0 & 1 & 0 \\ 1 & -3 & 0 \\ -\frac{3}{5} & -1+\frac{9}{5} & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -3 & 0 \\ -\frac{3}{5} & \frac{4}{5} & 1 \end{bmatrix}$

$L_3 \leftarrow 5 L_3.$

$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 5 & -2 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad$ $\begin{bmatrix} 0 & 1 & 0 \\ 1 & -3 & 0 \\ -3 & 4 & 5 \end{bmatrix}$

$L_1 \leftarrow L_1 - L_3$

$L_2 \leftarrow L_2 + 2L_3.$

$\begin{bmatrix} 1 & -2 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad L_2 \leftarrow L_2/5 \quad$ $\begin{bmatrix} 3 & -3 & -5 \\ -5 & 5 & 10 \\ -3 & 4 & 5 \end{bmatrix}$

$\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ $\quad$ $\begin{bmatrix} 3 & -3 & -5 \\ -1 & 1 & 2 \\ -3 & 4 & 5 \end{bmatrix}$

3. [15 points]

Consider the linear map represented by the matrix

$$A = \begin{pmatrix} 2 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 3 & 6 & 3 \end{pmatrix}$$

a) [7 points] Find a basis for the image of A (or equivalently the column space of A).

b) [8 points] Find a basis for the null-space of A.

a) $\begin{bmatrix} 2 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 3 & 6 & 3 \end{bmatrix}$ $\quad L_3 \leftarrow L_3 - 3L_2$

(*) $\begin{bmatrix} 2 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & -3 \end{bmatrix} \longrightarrow$ Basis :

$$\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$$

b) We know that the Null space has dimension

$5 - 3 = 2$

So it is enough to find to linearly independent vectors in the null space.

Since we easily see that $A \begin{pmatrix} 1 \\ -2 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \vec{0}$ and

$A \begin{pmatrix} 0 \\ 3 \\ -2 \\ 1 \\ 0 \end{pmatrix} = \vec{0}$, the vectors $\boxed{\begin{pmatrix} 1 \\ -2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ -2 \\ 1 \\ 0 \end{pmatrix}}$

give a basis for the Null space.

Alternatively, solve the system

(*) $\Longrightarrow$
$$\begin{aligned} x_5 &= 0 \\ x_3 + 2x_4 + 2x_5 &= 0 \\ 2x_1 + x_2 + 2x_3 + x_4 &= 0 \end{aligned}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}x_2 + \frac{3}{2}x_4 \\ x_2 \\ -2x_4 \\ x_4 \\ 0 \end{pmatrix} = x_2 \begin{pmatrix} -1/2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}$$

OR

4. [2 points each]
a) If $T : \mathbb{R}^6 \to \mathbb{R}^4$ is a linear map of rank 3 what is the dimension of the its null-space?
b) Suppose $A = (\mathbf{u}, \mathbf{v}, \mathbf{w})$ is a 3×3 matrix where $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are its columns. If $\det(A) = -5$ find $\det(2\mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{w}, 2\mathbf{w})$.
c) If $\det(A) = 2$ compute $\det(A^{-4})$.
d) For what values of x is the following matrix invertible

$$\begin{pmatrix} x & x & x \\ x & x & 1 \\ x & 1 & 1 \end{pmatrix}.$$

e) Write down a 3×3 matrix A with column space spanned by the vectors

$$\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

a) $\dim \text{Null} = 6 - \dim \text{Im}\, T = 6 - \text{rank}\, T = 3.$

b) $\det(2\vec{u} + \vec{v}, \vec{u} + \vec{w}, 2\vec{w})$
$= 2\det(2\vec{u} + \vec{v}, \vec{u} + \vec{w}, \vec{w})$
$= 2\det(2u + \vec{v}, \vec{u}, \vec{w}).$
$= 2\det(\vec{v}, \vec{u}, \vec{w})$
$= -2\det(\vec{u}, \vec{v}, \vec{w}) = -2 \times -5 = 10.$

c) $\det(A)^{-4} = (\det(A^{-1}))^4 = \left(\frac{1}{2}\right)^4 = \frac{1}{2^4} = \frac{1}{16}.$

d) $1 \begin{vmatrix} x & x & x \\ x & x & 1 \\ x & 1 & 1 \end{vmatrix} - x \begin{vmatrix} x & x \\ x & 1 \end{vmatrix} - 1 \begin{vmatrix} x & x \\ x & 1 \end{vmatrix} + 1 \begin{vmatrix} x & x \\ x & x \end{vmatrix}.$

$= x(x - x^2) - 1(x - x^2)$
$= x(1-x)\cancel{(x-1)}$ it is $\neq 0$ for $x \neq 0$ and $x \neq 1$

e) $\begin{bmatrix} 1 & 1 & 0 \\ 3 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix}$

5. [2 points each] For each of the following state whether the statement is TRUE or FALSE (no justification is necessary).

(a) For $n \times n$ matries A, B we have $\det(A+B) = \det(A) + \det(B)$.

(b) For **any** linear map $T : \mathbb{R}^m \to \mathbb{R}^n$ the image of T and null-space of T are both vector spaces.

(c) If A, B, C are square matrices and $AB = C$ and A is invertible then $B = CA^{-1}$.

(d) For a matrix A and scalar r we have $\det(rA) = r\det(A)$.

(e) The set of vectors of the form $\begin{bmatrix} t \\ 1 \end{bmatrix}$ is a subspace of $\mathbb{R}^2$.

a) FALSE ex $A = B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

b) TRUE

c) FALSE $B = A^{-1}C$ (is not equal to CA^{-1} in general. See review session before midterm exam)

d) FALSE $\det(rA) = r^n \det(A)$ when A has size n

e) FALSE (it does not contain $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$)