Solvability by radicals

We admit the following theorem :

Theorem. *Let $P \in \mathbb{Z}[X]$ a monic separable polynomial. Let p be a prime number and $\bar{P} \in \mathbb{F}_p[X]$ the reduction of P over $\mathbb{F}_p$. Suppose that $\bar{P}$ is separable, then the Galois group of P over $\mathbb{Q}$ contains a subgroup which is isomorphic to the Galois group of $\bar{P}$ over $\mathbb{F}_p$.*

We also admit that the symmetric group $\mathfrak{S}_5$ can be generated by a transposition and a 5-cycle.

***Problem* 1.** Let $n \geq 1$.

(1) Recall the definition of the alternating group $\mathfrak{A}_n$ of $\mathfrak{S}_n$. We recall that it is generated by the 3-cycles in $\mathfrak{S}_n$.
(2) Suppose from now on that $n \geq 5$. Let $\gamma = (a, b, c)$ be a 3-cycle in $\mathfrak{S}_n$. Let $d \neq e \in \{1, \ldots, n\} - \{a, b, c\}$ and $\sigma = (d, e)(b, c)$. Compute $\sigma\gamma\sigma^{-1}$.
(3) Show that $D(\mathfrak{A}_n) = \mathfrak{A}_n$ and that $\mathfrak{S}_n$ is not solvable.

***Problem* 2.** Let $P = X^5 - 5X^2 + 1$ with Galois group G over $\mathbb{Q}$.

(1) Show that G injects in $\mathfrak{S}_5$.
(2) Show that P has no root in $\mathbb{F}_2$ and $\mathbb{F}_4$, and that it is irreducible over $\mathbb{F}_2$.
(3) Show that P is irreducible over $\mathbb{Q}$ and deduce that 5 divides $|G|$.
(4) What is the Galois group of the reduction of P over $\mathbb{F}_2$? Deduce that G contains a 5 cycle.
(5) How many real roots does P have ? Show that the complex conjugation is an element in G.
(6) Deduce from the two previous questions that $G \simeq \mathfrak{S}_5$ and that the equation $P(x) = 0$ is not solvable by radicals.