Galois group of the algebraic closure of a finite field

Let q be a power of a prime number and $\overline{\mathbb{F}}_q$ a fixed algebraic closure of $\mathbb{F}_q$.

(1) For $n \geq 1$, recall the definition of the subfield $\mathbb{F}_{q^n}$ of $\overline{\mathbb{F}}_q$.

(2) Show that

$$\overline{\mathbb{F}}_q = \bigcup_{n \geq 1} \mathbb{F}_{q^n}$$

and that $\overline{\mathbb{F}}_q/\mathbb{F}_q$ is a Galois extension. Denote by G its Galois group. Let

$$\begin{aligned} F_q : \overline{\mathbb{F}}_q &\longrightarrow \overline{\mathbb{F}}_q \\ x &\longmapsto x^q \end{aligned}$$

(3) Show that $F_q \in G$.

(4) Describe the subgroup of G generated by F_q.

(5) For any $n \geq 1$, give a natural surjective morphism

$$\phi_n : G \longrightarrow \mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$$

such that

$$\bigcap_{n \geq 1} \ker(\phi_n)$$

is trivial.

(6) Denote by $\hat{\mathbb{Z}}$ the set of all sequences

$$(a_n)_{n \geq 1} \in \prod_{n \geq 1} \mathbb{Z}/n\mathbb{Z}$$

such that for any $m, n \geq 1$, the image of a_{mn} by the natural projection $\mathbb{Z}/mn\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ is a_n. Show that $\hat{\mathbb{Z}}$ is a group for the addition (defined coordinate by coordinate) and that $\hat{\mathbb{Z}}$ and G are isomorphic.

(7) Is G generated by F_q ?

(8) Show that an orbit $\mathcal{O}$ in $\overline{\mathbb{F}}_q$ under the action of G is finite. Show that for $x, x' \in \mathcal{O}$ we have $\mathbb{F}_q[x] = \mathbb{F}_q[x']$.

(9) Show that there is a surjection from the set of G-orbits in $\overline{\mathbb{F}}_q$ to the set of finite extensions of $\mathbb{F}_q$ contained in $\overline{\mathbb{F}}_q$. Is it a bijection ?