Review 2

Problem 1. (1) Prove that a group of order 4 is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/4\mathbb{Z}$.

(2) Find an extension of $\mathbb{Q}$ with Galois group $\mathbb{Z}/4\mathbb{Z}$.
(3) What is the Galois group of $X^8 - 1$ over $\mathbb{Q}$?
(4) What is the Galois group of $X^4 - 25$ over $\mathbb{Q}$?
(5) What is the Galois group of $X^4 + 4$ over $\mathbb{Q}$?

Problem 2. Find an extension E/F with Galois group $\mathbb{Z}/8\mathbb{Z}$.

Problem 3. Let $n \geq 1$. We want to show that there is a Galois extension of $\mathbb{Q}$ with Galois group $\mathbb{Z}/n\mathbb{Z}$.

(1) Let p be a prime number. Show that there is a Galois extension of $\mathbb{Q}$ with Galois group $\mathbb{Z}/(p-1)\mathbb{Z}$.
(2) Suppose that $p \equiv 1 \mod n$. Show that there is a Galois extension of $\mathbb{Q}$ with Galois group $\mathbb{Z}/n\mathbb{Z}$.
(3) By Dirichlet's theorem, given $a, b \geq 1$ with $gcd(a, b) = 1$, there are infinitely many primes of the form $a + mb$ where $m \geq 1$. Conclude the problem.

Problem 4. Find a Galois extension of $\mathbb{Q}$ of Galois group $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Problem 5. Let $n \in \mathbb{N}, n \geq 3$. Let R_n be the set of n^{th} roots of 1 in $\mathbb{C}$ represented in the complex plane as a regular n-gone. Let D_n be the group of isometries of $\mathbb{C}$ stabilizing R_n. This is the n^{th} dihedral group.

(1) Find a normal subgroup of D_n with order n.
(2) Show that $|D_n| = 2n$ and that $D_n \cong \mathbb{Z}/n\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$.
(3) What is the center of D_n ?

Problem 6. Let $P = X^4 - 2 \in \mathbb{Q}[X]$. Let K be the subfield of $\mathbb{C}$ generated over $\mathbb{Q}$ by the complex roots of P and G the Galois group of P over $\mathbb{Q}$. Let $x := 2^{1/4}$.

(1) Show that $L := \mathbb{Q}[x]$ is not a Galois extension of Q.
(2) Is G a commutative group ?
(3) Compute $[L : \mathbb{Q}]$.
(4) Show that $G = \mathbb{Q}[x, i]$ and $[K : \mathbb{Q}] = 8$.
(5) Let C be the set of roots of P in $\mathbb{C}$. Draw C in the complex plane. Show that the action of G on C preserves the distances that is to say, for $c, c' \in C$ we have $|g(c) - g(c')| = |c - c'|$.
(6) Show that $G \cong D_4$.
(7) Show that there is a unique element r in G such that $r(x) = ix$ and $r(i) = i$. What is the order of r in G ?
(8) Show that there is a unique element s in G such that $s(x) = x$ and $s(i) = -i$. What is the order of s in G ?
(9) Show that $srs^{-1} = r^{-1}$.
(10) Show that s and r generate G (Use Problem 2).

(11) What can you say about $x+i$?

Remark. Let G be a group of order 8. By splitting into cases depending on the maximum order of an element in G, one can show that G is isomorphic to one of the following groups :

$$\mathbb{Z}/8\mathbb{Z},\ ,(\mathbb{Z}/2\mathbb{Z})^3,\ \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z},\ D_4, \mathbb{H}_8$$

where $\mathbb{H}_8 = \{\pm 1, \pm i, \pm j, \pm k\}$is the group of quaternions given by

$$\begin{array}{ll} i^2 = j^2 = k^2 = -1, & ij = -ji = k, \\ jk = -kj = i, & ki = -ik = j. \end{array}$$

One can show that $\mathbb{H}_8$ cannot be written as a semi direct product of 2 of its strict subgroups. One can show that $\mathbb{Q}(\sqrt{(2+\sqrt{2})(3+\sqrt{6})})/\mathbb{Q}$ is Galois with Galois group $\mathbb{H}_8$. Therefore any group of order 8 can be realized as the Galois group of an extension of $\mathbb{Q}$.