Review

Problem 1. Let $x := 2^{1/3}$ and $K := \mathbb{Q}(x, j)$.

(1) Show that $[K : \mathbb{Q}] = 6$.

(2) Show that $|\mathrm{Aut}_{\mathbb{Q}}(K)| \leq 6$.

(3) Compute $\mathrm{Aut}_{\mathbb{Q}(x)}(K)$ and $\mathrm{Aut}_{\mathbb{Q}(j)}(K)$ and prove that $|\mathrm{Aut}_{\mathbb{Q}}(K)| = 6$. (One can also prove this by first showing that $K/\mathbb{Q}$ is Galois).

(4) Let $R := \{x, jx, j^2x\}$. Show that any $\sigma \in \mathrm{Aut}_{\mathbb{Q}(x)}(K)$ induces a permutation of R and that $\mathrm{Aut}_{\mathbb{Q}(x)}(K) \simeq \mathfrak{S}_3$.

Problem 2. Let k be a perfect field and K/k a finite extension.

(1) Recall the definition of a perfect field.

(2) Recall what it means for the extension K/k to be a Galois extension and recall the definition of $\mathrm{Gal}(K/k)$.

(3) Suppose that K/k is a Galois extension. Show that $[K : k] = |\mathrm{Gal}(K/k)|$ and that the set $K^{\mathrm{Gal}(K/k)}$ of $\mathrm{Gal}(K/k)$-fixed points in K is k.

Problem 3. Show that $\mathbb{Q}[\sqrt{2+\sqrt{2}}]$ is a Galois extension of $\mathbb{Q}$.

Problem 4. Let k be a field. Let $\bar{k}$ denote an algebraic closure of k.

(1) Let $x \in \bar{k}$. Show that the set $\mathrm{conj}_k(x)$ of k-conjugates of x in $\bar{x}$ is

$$\{\sigma(x), \sigma \in \mathrm{Aut}_k(\bar{k})\} = \{\sigma(x), \sigma \in \mathrm{Hom}_k(K, \bar{k})\}$$

where $k \subseteq K \subset \bar{k}$ is any extension containing x.

(2) For $x, y \in \bar{k}$, show that $\mathrm{conj}_k(x+y) \subseteq \mathrm{conj}_k(x) + \mathrm{conj}_k(y)$ and give an example where this inclusion is not an equality.

(3) Give and example where $\mathrm{conj}_k(x)$ is not contained in $k[x]$.

(4) Suppose that k is perfect. Show that the following are equivalent :

   (a) $x \in k$;

   (b) $\mathrm{conj}_k(x) = \{x\}$;

   (c) for any $\sigma \in \mathrm{Aut}_k(\bar{k})$ we have $\sigma(x) = x$.

Problem 5 (Artin's Lemma)**.** Let K be a perfect field and G a finite subgroup of the group of automorphisms of the field K.

(1) Show that $k := K^G$ is perfect and check that G is a subgroup of $\mathrm{Aut}_k(K)$.

(2) Let $x \in G$ and Gx its orbit under the action of G.

   (a) Show that $|Gx| \leq |G|$.

   (b) Show that x is algebraic over k with degree $\leq |G|$.

(3) Let $x \in K$ with maximal degree over k. Show that x is a primitive element for K and that $[K : k] \leq |G|$.

(4) Show that $G = \mathrm{Aut}_k(K)$, that $[K : k] = |G|$ and that K/k is a Galois extension.

Problem 6. Let k be a field and $n \geq 1$.

(1) Show that $\mathfrak{S}_n$ can be considered as a group of automorphisms of the field $k(X_1, ..., X_n)$.

(2) Let G be a group with n elements. Use Artin's lemma to find a Galois extension with Galois group G.