Homework 5

Problem 1. Let G be a finite group. Show that G is solvable if and only if there is a sequence of groups

$$\{1\} = G_0 \subset G_1 \subset \cdots \subset G_n = G$$

such that G_i is a normal subgroup of G_{i+1} and G_{i+1}/G_i is cyclic for any $i = 0, \dots n-1$.

Problem 2. Let $P \in k[X]$ be a separable polynomial with degree n and G its Galois group over k. We see G as a subgroup of $\mathfrak{S}_n$. Let $\mathcal{R}$ be the set of roots of P in an algebraic closure Ω of k. Let

$$\mathrm{disc}(P) = (-1)^{n(n-1)/2} \prod_{x,y \in \mathcal{R}, x \neq y} (x - y).$$

(1) Why do we have $\mathrm{disc}(P) \in k^{\times}$?

(2) Give an explicit formula for a $\delta \in \Omega$ such that $\delta^2 = \mathrm{disc}(P)$ and such that $\sigma(\delta) = \epsilon(\sigma)\delta$ of any $\sigma \in G$ (and where $\epsilon : \mathfrak{S}_n \to \{\pm 1\}$ denotes the signature).

(3) Show that if $G \subset \mathfrak{A}_n$ then $\delta \in k$.

(4) Show that if k has characteristic different from 2 and $\delta \in k$ then $G \subset \mathfrak{A}_n$.

Problem 3. (1) Show that the Galois group of a polynomial with degree 2 is trivial or $\mathbb{Z}/2\mathbb{Z}$.

(2) Show that the Galois group of an irreducible polynomial $P \in k[X]$ with degree 3 is either $\mathfrak{A}_3$ or $\mathfrak{S}_3$.

(3) Show that the Galois group of $X^3 - 2$ over $\mathbb{Q}$ is $\mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$.

Problem 4. Let p be a prime number, Ω an algebraic closure of $\mathbb{F}_p$ and $r \geq 1$. Let $P_1, \dots P_r$ be r irreducible polynomials in $\mathbb{F}_p[X]$ with respective degrees $d_1 \dots d_r$. We suppose that they are pairwise coprime.
Let K be the subfield of Ω generated by $\mathbb{F}_p$ and the roots of all P_is for $1 \leq i \leq r$.

(1) Show that the Frobenius of K is a product of r cycles with disjoint support and respective lengths $d_1 \dots d_r$.

(2) Deduce that the Galois group of K has cardinality the LCM of $d_1, \dot{;} d_r$.