Homework 3

I suggest that you work on Problems 1, 2, 3 and 5 for the homework, + Problem 7 of HW1.

Problem 1. Let K/k be an algebraic extension. We want to show that any element in $\mathrm{Hom}_k(K,K)$ in an element in $\mathrm{Aut}_k(K)$.

(1) Prove the claim when K/k is finite.

(2) Deduce the general case. *Hint : For $x \in K$, consider the sub-k-algebra of K generated by the roots of Π_x in K.*

Problem 2. Let K/k and Ω/k two fields extension. Suppose that K/k is algebraic and that Ω is algebraically closed.

(1) Let E be the set of pairs (L,σ) where L is a subfield of K containing k (or rather the image of k in K) and $\sigma : L \to \Omega$ a k-embedding.

  (a) Show that E is not empty.

  (b) Describe on E a natural partial order.

  (c) Show that every totally ordered subset of E has an upper bound. Apply Zorn's lemma and deduce that E has (at least) one maximal element (L_0, σ_0).

  (d) Show that $L_0 = K$. *Otherwise, there is $x \in K - L_0$. It is algebraic over L and consider $L[X]/\Pi_{x,L}$...*

  (e) State the theorem that you just proved.

(2) Show that two algebraic closures K_1 and K_2 of a field k_0 are isomorphic.

(3) Let $x \in K$. Show that the set of k-conjugates of x in Ω is the set of all $\sigma(x)$ for $\sigma \in \mathrm{Hom}_k(K,\Omega)$.

Problem 3. Let $\overline{\mathbb{Q}} \subset \mathbb{C}$ be the subfield of the elements which are algebraic over $\mathbb{Q}$.

(1) Recall why $\overline{\mathbb{Q}}$ is an algebraic closure for $\mathbb{Q}$.

(2) Show that $\mathrm{Hom}_{\mathbb{Q}}(\overline{\mathbb{Q}},\mathbb{C}) = \mathrm{Hom}_{\mathbb{Q}}(\overline{\mathbb{Q}},\overline{\mathbb{Q}}) = \mathrm{Aut}_{\mathbb{Q}}(\overline{\mathbb{Q}})$.

(3) Show that $\overline{\mathbb{Q}}$ is countable.

(4) Show that if $\sigma \in \mathrm{Aut}_{\mathbb{Q}}(\overline{\mathbb{Q}})$ is continuous (for the topology induced by the usual topology of $\mathbb{C}$) then σ is either the identity of the complex conjugation.

(5) Let $n \geq 1$ and ζ a n^{th} root of 1. Show that there is an element $\sigma \in \mathrm{Aut}_{\mathbb{Q}}(\overline{\mathbb{Q}})$ such that $\sigma(\sqrt[n]{2}) = \zeta\sqrt[n]{2}$.

(6) Show that $\mathrm{Aut}_{\mathbb{Q}}(\overline{\mathbb{Q}})$ is infinite.

Problem 4. Let k be a field and $P \in k[X]$.

(1) Suppose that P irreducible. Let $k \subseteq L$ be an extension and suppose that L contains a root for P. Show that there is a k-morphism from the stem field of P into L.

(2) Suppose that P is not constant. Show by induction on the degree n of P that there is an extension L/k with degree $\leq n!$ such that P splits in L.

Problem 5. We admit the following (deep) result :
Let k be a field and A a finitely generated k-algebra. If A is a field then it is finite dimensional over k.
Show that the maximal ideals of the polynomial algebra $\mathbb{C}[X_1, ..., X_n]$ are the ideals of the form

$$\langle X_1 - x_1, \dots, X_n - x_n \rangle$$

for $(x_1, ..., x_n) \in \mathbb{C}^n$.

Problem 6 (Ruler and Compass Constructions)**.** Armed with a straightedge, a compass and two points 0 and 1 marked on an otherwise blank plane, the game is to see which complex numbers (or points of $\mathbb{R}^2$) you can construct, and which complex numbers you cannot construct.

Definition. A point p is constructible if $p = (0, 0)$ or $p = (1, 0)$ or else p is an intersection point of a pair of lines, a line and a circle, or a pair of circles that you can draw with your straightedge and compass.

The Rules : With your straightedge and compass, you are allowed to :
(i) Draw the line $L(p, q)$ (with the straightedge) through any two points p and q that you have already constructed.
(ii) Open the compass to span the distance $|q - p|$ between any two points p and q that you have already constructed, place the base at a third point ω (already constructed), and draw the circle $C(\omega; |q - p|)$.

(1) Show that the set $\mathcal{C}$ of all coordinates of the constructible points is a subfield of $\mathbb{R}$ containing $\mathbb{Q}$.

(2) Show that for $x > 0$, if $x \in \mathcal{C}$, then $\sqrt{x} \in \mathcal{C}$.

Let k be a subfield of $\mathbb{R}$.

(3) Show that a line $L \subset \mathbb{R}^2$ containing two distinct points in k^2 has an equation of the form $ax^2 + bx + c = 0$ with $a, b, c \in k$. We say that L is defined over k.

(4) Show that a circle $C \subset \mathbb{R}^2$ with center in k^2 and containing a point in k^2 has an equation of the form $x^2 + y^2 + dx + ey + f = 0$ with $d, e, f \in k$. We say that C is defined over k.

(5) Suppose that a point $(x, y) \in \mathbb{R}^2$ is contained in the intersection of
   – two distinct lines defined over k, or
   – two distinct circles defined over k, or
   – a line and a circle both defined over k.
   Show that $[k(x) : k] \leq 2$ and $[k(y) : k] \leq 2$.

(6) Let $(x, y) \in \mathbb{R}^2$.

(a) Show that the point (x, y) is constructible (see the definition above) if and only if x and y lie in $\mathcal{C}$.

(b) Show that the point (x, y) is constructible if and only if there is a finite tower of subfields of $\mathbb{R}$

$$\mathbb{Q} = k_0 \subset k_1 \subset \cdots \subset k_N \subset \mathbb{R}$$

with $N \geq 0$ such that $[k_{i+1} : k_i] = 2$ for any $i < N$ such that $(x, y) \in k_N^2$.

(7) Prove that any $x \in \mathcal{C}$ is algebraic (over $\mathbb{Q}$) of degree a power of 2.

(8) Is the regular 7-gon constructible? Justify.