Algebraicity

Problem 1. Let $\alpha = \sqrt[3]{7}$. We consider the map

$$\begin{aligned} f_\alpha : \mathbb{Q}[X] &\longrightarrow \mathbb{C} \\ P &\longmapsto P(\alpha) \end{aligned}$$

(1) Show that f_α is a morphism of $\mathbb{Q}$-algebras.

(2) Show that it is not injective and describe its kernel.

(3) Show that its image is a finite dimensional $\mathbb{Q}$-subalgebra of $\mathbb{C}$.

(4) Find $U, V \in \mathbb{Q}[X]$ such that $XU + (X^3 - 7)V = 1$.

(5) What can you say about the element α in the ring $\mathrm{Im}(f_\alpha)$?

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More generally, to $\alpha \in \mathbb{C}$ we attach the morphism of $\mathbb{Q}$-algebras

$$\begin{aligned} f_\alpha : \mathbb{Q}[X] &\longrightarrow \mathbb{C} \\ P &\longmapsto P(\alpha) \end{aligned}$$

and denote by $\mathbb{Q}[\alpha]$ its image. By definition, we have

$$\mathbb{Q}[\alpha] = \{P(\alpha), P \in \mathbb{Q}[X]\}.$$

Definition. *Let $\alpha \in \mathbb{C}$, we say that α is algebraic over $\mathbb{Q}$ if f_α is not injective, in which case :*

– *we denote by Π_α the unique unitary generator of* $\ker(f_\alpha)$. *It is the* minimal polynomial of α.

– *the degree of α (over $\mathbb{Q}$) is by definition the degree of* Π_α

***Problem* 2.** Let $\alpha \in \mathbb{C}$ be algebraic over $\mathbb{Q}$. Show that $\Pi_\alpha \in \mathbb{Q}[X]$ is the unique unitary irreducible polynomial in $\ker(f_\alpha)$.

Prove the following theorem

Theorem. *Let $\alpha \in \mathbb{C}$. The following are equivalent :*

(1) *α is algebraic over $\mathbb{Q}$.*
(2) *The $\mathbb{Q}$-algebra $\mathbb{Q}[\alpha]$ is finite dimensional over $\mathbb{Q}$.*
(3) *The $\mathbb{Q}$-algebra $\mathbb{Q}[\alpha]$ is a field.*

and give a method, in the case when α is algebraic over $\mathbb{Q}$, to find the inverse in $\mathbb{Q}[\alpha]$ of a nonzero element.