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Let k be a field. Let A be a k-algebra. We recall that k is an injective module over itself.

Let Y be a left A-module. Then $Y^\vee := \mathrm{Hom}_k(Y,k)$ is naturally a right A-module. Let X be a right A-module, we recall the following standard isomorphism of vector spaces (which you can check) :

$$\mathrm{Hom}_A(X, Y^\vee) \cong \mathrm{Hom}_k(X \otimes_A Y, k).$$

(1) Show that if Y is a projective left A-module then $Y^\vee$ is an injective right A-module.

(2) Suppose now that A is finite dimensional over k. It is a Frobenius algebra if there is a linear map $\lambda : A \to k$ the kernel of which does not contain any nonzero left or right ideal of A.

  (a) Show that the group algebra of a finite group is a Frobenius algebra.

  (b) Suppose that A is a Frobenius algebra. Here we denote by ${}_AA$ (resp. A_A)the space A with the structure of left (resp. right) module over itself.

    (i) Show that the map ${}_AA \to (A_A)^\vee$, $a \mapsto (b \mapsto \lambda(ba))$ is an isomorphism of left A-modules.

    (ii) Show that ${}_AA$ is an injective module. We say that A is self-injective.

    (iii) Show that a principal indecomposable A-module is injective.

(3) We consider the algebra $A := k[B\backslash \mathrm{GL}_3(\mathbb{F}_q)/B]$ where q is a power of a prime number p. It is generated by S_1 and S_2 with the relations

$$S_i^2 = (q-1)S_i + q \text{ for } i = 1,2 \text{ and } S_1S_2S_1 = S_2S_1S_2.$$

We set $S_i^* := S_i + 1 - q$.

  (a) We suppose that k has characteristic p.

    (i) Compute $(S_i^*)^2$.

    (ii) What are the 1-dimensional A-modules ?

    (iii) Let $X := -S_1S_2S_1$, $Y := S_1S_2^*S_1$, $Z := -S_1^*S_2S_1^*$, $T := S_1^*S_2^*S_1^*$. Show that they are indecomposable idempotents of A. Show that X, T and $Y + Z$ are central idempotents.

    (iv) Show that all the simple A-modules are 1 dimensional and describe the composition series of ${}_AA$.

    (v) Show that 2 of the simple A-modules are projective and that the other have infinite projective dimension.

    (vi) Describe a basis for A and show that A is a Frobenius algebra (consider the linear map attaching to an element the coefficient of $S_1S_2S_1$...)

  (b) Suppose that k has characteristic different from p.

    (i) What are the 1-dimensional A-modules ? One can show that they are projective. (Introducing the right idempotents : $T := 1+S_1+S_2+S_1S_2+S_2S_1+S_1S_2S_1$ and $X := q^3 - q^2S_1 - q^2S_2 + qS_1S_2 + qS_2S_1 - S_1S_2S_1$). This is true more generally for $k[B\backslash \mathrm{GL}_n(\mathbb{F}_q)/B]$.

    (ii) Show that A has 2 non-isomorphic 2-dimensional simple modules and that A is semisimple.

It is a general fact that a Frobenius algebra is either semisimple or it has infinite global dimension. Illustrate this fact with the group algebra of a finite cyclic group.