

Math 221, Term 2, 2017–2018

Instructor: R. Ollivier (Section 202)

Thursday, February 8, 2018

Instructions

- This is a closed book/notes exam. Use of calculators is not permitted.
- You have 50 minutes.
- Please do all your work on the exam provided.
- You must show your work to receive full credit on a problem (except for the multiple choice questions).

Print name: _____

Print student #: _____

Signature: _____

Grader's use only:

1. _____ /15

2. _____ /10

3. _____ /10

4. _____ /10

5. _____ /10

1. [15 points] Find the general solution of the following system of equations.
Write your answer in parametric form.

$$\begin{aligned} 5x_2 + x_3 &= 2 \\ x_1 - x_2 + 2x_3 + x_4 &= 1 \\ 3x_1 - 8x_2 + 5x_3 + 3x_4 &= 1 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 0 & 5 & 1 & 0 & 2 \\ 1 & -1 & 2 & 1 & 1 \\ 3 & -8 & 5 & 3 & 1 \end{array} \right]$$

$$L_1 \leftrightarrow L_2$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & 1 & 1 \\ 0 & 5 & 1 & 0 & 2 \\ 3 & -8 & 5 & 3 & 1 \end{array} \right]$$

$$L_3 \leftarrow L_3 - 3L_1$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & 1 & 1 \\ 0 & 5 & 1 & 0 & 2 \\ 0 & -5 & -1 & 0 & -2 \end{array} \right]$$

$$L_3 \leftarrow L_3 + L_2$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & 1 & 1 \\ 0 & 5 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

$$\text{So } x_1 = 1 + x_2 - 2x_3 - x_4$$

$$5x_2 = 2 - x_3$$

$$\begin{aligned} \rightarrow x_2 &= 1 - 2x_3 - x_4 + \frac{2}{5} - \frac{x_3}{5} \\ &= \frac{7}{5} - \frac{11}{5}x_3 - x_4 \end{aligned}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 7/5 \\ 7/5 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -11/5 \\ -1/5 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

2. [10 points]

a) [4 points] Are the following 3 vectors linearly independent? Explain why or why not.

$$v_1 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 5 \\ 2 \end{bmatrix}, v_3 = \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix}.$$

b) [3 points] Is the vector $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ in the span v_1, v_2 and v_3 ? Explain.

c) [3 points] Let $A = \begin{pmatrix} 2 & -1 & 3 \\ 3 & 5 & -2 \\ 4 & 2 & 2 \end{pmatrix}$.

Find a nonzero vector x such that $Ax = 0$. Is A one-to-one?

a) No because $v_3 = v_1 - v_2$

b) Solve $\left[\begin{array}{ccc|c} 2 & -1 & 3 & 0 \\ 3 & 5 & -2 & 1 \\ 4 & 2 & 2 & 0 \end{array} \right]$

$$L_2 \leftarrow L_2 - \frac{3}{2}L_1$$

$$L_3 \leftarrow L_3 - 2L_2$$

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 0 \\ 0 & 13/2 & -13/2 & 1 \\ 0 & 4 & -4 & -2 \end{array} \right]$$

$$L_2 \leftarrow L_2 \times 2/13$$

$$L_3 \leftarrow L_3/4$$

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 0 \\ 0 & 1 & -1 & 2/13 \\ 0 & 1 & -1 & -1/2 \end{array} \right]$$

$$L_3 \leftarrow L_3 - L_2 \left[\begin{array}{ccc|c} 2 & -1 & 3 & 0 \\ 0 & 1 & -1 & 2/13 \\ 0 & 0 & 0 & -1/2 - 2/13 \end{array} \right]$$

inconsistent system so the answer is NO.

c) Since $v_3 = v_1 - v_2$

we have

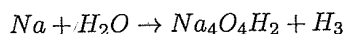
$$A \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \vec{0}$$

So $x = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ works.

A is not ONE TO ONE



3. [10 points] Balance the chemical equation



and give your answer in lowest terms.

Solve

$$x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = x_3 \begin{bmatrix} 4 \\ 2 \\ 4 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -4 & 0 \\ 0 & 2 & -2 & -3 \\ 0 & 1 & -4 & 0 \end{bmatrix} \xrightarrow{L_2 \leftarrow L_2 - 2L_3} \begin{bmatrix} 1 & 0 & -4 & 0 \\ 0 & 0 & 6 & -3 \\ 0 & 1 & -4 & 0 \end{bmatrix}$$

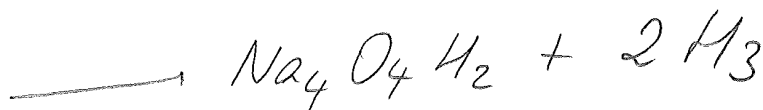
$$\begin{matrix} \leftarrow L_2 \leftrightarrow L_3 \\ \begin{bmatrix} 1 & 0 & -4 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 6 & -3 \end{bmatrix} \end{matrix}$$

$$2x_3 = x_4$$

$$x_2 = 4x_3$$

$$x_1 = 4x_3$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} 4 \\ 4 \\ 1 \\ 2 \end{pmatrix}$$



4. [10 points]

Find the 2×2 matrices which describe the following linear maps.

b) [3 points] rotation by 90 degrees clockwise.

b) [3 points] projection on the x -axis.

c) [4 points] reflection about the origin (the point $(0,0)$).

$$a) \begin{aligned} T(e_1) &= -e_2 \\ T(e_2) &= e_1 \end{aligned} \longrightarrow \text{matrix } \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$b) \begin{aligned} T(e_1) &= e_1 \\ T(e_2) &= 0 \end{aligned} \text{ matrix } \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$c) \begin{aligned} T(e_1) &= -e_1 \\ T(e_2) &= -e_2 \end{aligned} \text{ matrix } \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

5. [10 points] For each of the following state whether the statement is TRUE or FALSE (no justification is necessary).

- (a) The map $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ given by $(x, y) \mapsto (x + y + 1, 2y, x)$ is linear.
- (b) If vectors v_1, v_2 and v_3 in \mathbb{R}^3 are linearly independent then so must be the vectors v_1 and v_2 .
- (c) Given three vectors v_1, v_2, v_3 in \mathbb{R}^4 which are linearly independent and A the matrix with columns v_1, v_2 , and v_3 , the map $x \mapsto Ax$ is onto.
- (d) A linear map $\mathbb{R}^3 \rightarrow \mathbb{R}^2$ must be onto.
- (e) If x_1 and x_2 are solutions to $Ax = b$ then $x_1 - x_2$ is also a solution.

(a) no, $T(0, 0) \neq (0, 0, 0)$

(b) yes

(c) No ex

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

(d) No ex: $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$
 $\vec{x} \mapsto \vec{0}$

(e) No