Math 221, Term 2, 2017–2018

Instructor: R. Ollivier (Section 202)

Thursday, February 8, 2018

Instructions

- This is a closed book/notes exam. Use of calculators is not permitted.
- You have **50 minutes.**
- Please do all your work on the exam provided.
- You must show your work to receive full credit on a problem (except for the multiple choice questions).

Print name: ______________________________

Print student #: ______________________________

Signature: ______________________________

Grader's use only:

1. ________ /15

2. ________ /10

3. ________ /10

4. ________ /10

5. ________ /10

1. [15 points] Find the general solution of the following system of equations. Write your answer in parametric form.

$$5x_2 + x_3 = 2$$
$$x_1 - x_2 + 2x_3 + x_4 = 1$$
$$3x_1 - 8x_2 + 5x_3 + 3x_4 = 1$$

$$\left[\begin{array}{cccc|c} 0 & 5 & 1 & 0 & 2 \\ 1 & -1 & 2 & 1 & 1 \\ 3 & -8 & 5 & 3 & 1 \end{array}\right]$$

$L_1 \leftrightarrow L_2$

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & 1 & 1 \\ 0 & 5 & 1 & 0 & 2 \\ 3 & -8 & 5 & 3 & 1 \end{array}\right]$$

$L_3 \leftarrow L_3 - 3L_1$

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & 1 & 1 \\ 0 & 5 & 1 & 0 & 2 \\ 0 & -5 & -1 & 0 & -2 \end{array}\right]$$

$L_3 \leftarrow L_3 + L_2$

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & 1 & 1 \\ 0 & 5 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

So $x_1 = 1 + x_2 - 2x_3 - x_4$

$5x_2 = 2 - x_3$.

$x_1 = 1 - 2x_3 - x_4 + \frac{2}{5} - \frac{x_3}{5}$

$= \frac{7}{5} - \frac{11}{5}x_3 - x_4$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 7/5 \\ 2/5 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -11/5 \\ -1/5 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

2. [10 points]
a) [4 points] Are the following 3 vectors linearly independent? Explain why or why not.

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1 \\ 5 \\ 2 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix}.$$

b) [3 points] Is the vector $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ in the span $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$? Explain.

c) [3 points] Let $A = \begin{pmatrix} 2 & -1 & 3 \\ 3 & 5 & -2 \\ 4 & 2 & 2 \end{pmatrix}$.
Find a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = 0$. Is A one-to-one?

a) No because $\vec{V_3} = \vec{V_1} - \vec{V_2}$

b) Solve $\left[\begin{array}{ccc|c} 2 & -1 & 3 & 0 \\ 3 & 5 & -2 & 1 \\ 4 & 2 & 2 & 0 \end{array}\right]$

$L_2 \leftarrow L_2 - \frac{3}{2} L_1$

$L_3 \leftarrow L_3 - 2L_2$

$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 0 \\ 0 & 13/2 & -13/2 & 1 \\ 0 & 4 & -4 & -2 \end{array}\right]$

$L_2 \leftarrow L_2 \times 2/13$

$L_3 \leftarrow L_3 / 4$

$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 0 \\ 0 & 1 & -1 & 2/13 \\ 0 & 1 & -1 & -1/2 \end{array}\right]$

$L_3 \leftarrow L_3 - L_2$ $\left[\begin{array}{ccc|c} 2 & -1 & 3 & 0 \\ 0 & 1 & -1 & 2/13 \\ 0 & 0 & 0 & -\frac{1}{2} - 2/3 \end{array}\right]$

INCONSISTENT SYSTEM So the answer is NO.

c) Since $\vec{V_3} = \vec{V_1} - \vec{V_2}$ we have

$A \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = \vec{0}$

So $\vec{x} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$ works.

A is not ONE TO ONE.

$$x_1 Na + x_2 H_2 O \longrightarrow x_3 Na_4 O_4 H_2 + x_4 H_3$$

3. [10 points] Balance the chemical equation

$$Na + H_2O \rightarrow Na_4O_4H_2 + H_3$$

and give your answer in lowest terms.

Solve
$$x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = x_3 \begin{bmatrix} 4 \\ 2 \\ 4 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 0 & -4 & 0 \\ 0 & 2 & -2 & -3 \\ 0 & 1 & -4 & 0 \end{bmatrix} \xrightarrow{L_2 \leftarrow L_2 - 2L_3} \begin{bmatrix} 1 & 0 & -4 & 0 \\ 0 & 0 & 6 & -3 \\ 0 & 1 & -4 & 0 \end{bmatrix}$$

$$\xrightarrow{L_2 \leftrightarrow L_3} \begin{bmatrix} 1 & 0 & -4 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 6 & -3 \end{bmatrix}.$$

$$2x_3 = x_4$$
$$x_2 = 4x_3$$
$$x_1 = 4x_3.$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} 4 \\ 4 \\ 1 \\ 2 \end{pmatrix}$$

$$4 Na + 4 H_2 O \longrightarrow Na_4 O_4 H_2 + 2 H_3.$$

4. [10 points]
Find the 2×2 matrices which describe the following linear maps.
b) [3 points] rotation by 90 degrees clockwise.
b) [3 points] projection on the x-axis.
c) [4 points] reflection about the origin (the point $(0,0)$).

a) $T(e_1) = -e_2$
$T(e_2) = e_1$ $\longrightarrow$ matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

b) $T(e_1) = e_1$
$T(e_2) = 0$ matrix $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

c) $T(e_1) = -e_1$
$T(e_2) = -e_2$ matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

5. [10 points] For each of the following state whether the statement is TRUE or FALSE (no justification is necessary).

(a) The map $T : \mathbb{R}^2 \to \mathbb{R}^3$ given by $(x, y) \mapsto (x + y + 1, 2y, x)$ is linear.

(b) If vectors $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ in $\mathbb{R}^3$ are linearly independent then so must be the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$.

(c) Given three vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ in $\mathbb{R}^4$ which are linearly independent and A the matrix with columns $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$, the map $\mathbf{x} \mapsto A\mathbf{x}$ is onto.

(d) A linear map $\mathbb{R}^3 \to \mathbb{R}^2$ must be onto.

(e) If $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions to $A\mathbf{x} = \mathbf{b}$ then $\mathbf{x}_1 - \mathbf{x}_2$ is also a solution.

(a) no, $T(0,0) \neq (0,0,0)$

(b) yes.

(c) No <u>ex</u> $\vec{v_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $\vec{v_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ $\vec{v_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

(d) No <u>ex:</u> $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$, $\vec{x} \longmapsto \vec{0}$.

(e) No