ANOMALOUS THRESHOLD BEHAVIOR OF LONG RANGE RANDOM WALKS.

MATHAV MURUGAN AND LAURENT SALOFF-COSTE†

Abstract. We consider weighted graphs satisfying sub-Gaussian estimate for the natural random walk. On such graphs, we study symmetric Markov chains with heavy tailed jumps. We establish a threshold behavior of such Markov chains when the index governing the tail heaviness (or jump index) equals the escape time exponent (or walk dimension) of the sub-Gaussian estimate. In a certain sense, this generalizes the classical threshold corresponding to the second moment condition.

1. Introduction

This work concerns a new threshold behavior of random walks on graphs driven by low moment measures. As the title suggests, this work combines two lines of research that have been actively pursued: anomalous random walks and long range random walks. The graphs were are interested in have a nearest neighbor random walk that satisfies sub-Gaussian estimates. Sub-Gaussian estimates for nearest neighbor random walks are typical of many regular fractals like Sierpinski gaskets, carpets and the Viscek graphs. See [22] for a recent survey on such *anomalous random walks*. Another line of work that has received much attention recently is the long term behavior of random walks with *heavy tailed jumps*. For example [5], [10], [11], [2], [4] are just a few works in this direction. In much of the existing literature the 'jump index' β is assumed to be in $(0, 2)$. Our work is a modest attempt to understand the behavior of such random walks when $\beta \in (0, \infty)$.

The motivation for our work comes from a recent work by the second author and Zheng [24]. In [24], the behavior of long range random walks on groups was investigated for the full range of the jump index $\beta \in (0, \infty)$. For random walks on groups there is a threshold behavior at $\beta = 2$. For graphs satisfying a sub-Gaussian heat kernel estimate, we show that the threshold behavior happens when the jump index β equals the escape time exponent.

Let Γ be an infinite, connected, locally finite graph endowed with a *weight* μ_{xy}. The elements of the set Γ are called vertices. Some of the vertices are connected by an edge, in which case we say that they are neighbors. The weight is a symmetric non-negative function on $\Gamma \times \Gamma$ such that $\mu_{xy} > 0$ if and only if x and y are neighbors (in which case we write $x \sim y$). We call the pair (Γ, μ) a *weighted graph*.

Date: June 25, 2015.

2010 *Mathematics Subject Classification.* 60J10, 60J75, 60J15.

†Both the authors were partially supported by NSF grant DMS 1404435.

The weight μ_{xy} on the edges induces a weight $\mu(x)$ on the vertices and a measure μ on subsets $A \subset \Gamma$ defined by

$$\mu(x) := \sum_{y:y\sim x} \mu_{xy} \quad \text{and} \quad \mu(A) := \sum_{x\in A} \mu(x).$$

Let $d(x,y)$ be the graph distance between points $x, y \in \Gamma$, that is the minimal number of edges in any edge path connecting x and y. Denote the metric balls and their measures as follows

$$B(x,r) := \{y \in \Gamma : d(x,y) \leq r\} \quad \text{and} \quad V_\mu(x,r) := \mu(B(x,r))$$

for all $x \in \Gamma$ and $r \geq 0$. We assume that the measure μ is comparable to the counting measure in the sense that there exists $C_\mu \in [1,\infty)$ such that $\mu_x = \mu(\{x\})$ satisfies

$$C_\mu^{-1} \leq \mu_x \leq C_\mu \tag{1}$$

We consider weighted graphs (Γ,μ) satisfying the following uniform volume doubling assumption: there exists $V : [0,\infty) \to (0,\infty)$, a strictly increasing continuous function and constants $C_D, C_h > 1$ such that

$$V(2r) \leq C_D V(r) \tag{2}$$

for all $r > 0$ and

$$C_h^{-1} V(r) \leq V_\mu(x,r) \leq C_h V(r) \tag{3}$$

for all $x \in \Gamma$ and for all $r > 0$. It can be easily seen from (2) that

$$\frac{V(R)}{V(r)} \leq C_D \left(\frac{R}{r}\right)^\alpha \tag{4}$$

for all $0 < r \leq R$ and for all $\alpha \geq \log_2 C_D$. For the rest of the work, we will assume that our weighted graph (Γ,μ) satisfies (1), (2) and (3).

Remark. If (Γ,μ) satisfies (2) and (3), we may assume that $V(n) = V_\mu(x_0,n)$ for some fixed x_0 and for all natural numbers n. For non-integer values we can extend it by linear interpolation. Since the graph is connected, infinite and locally finite, the function V defined above is continuous, strictly increasing on $[0,\infty)$.

There is a natural random walk X_n on (Γ,μ) associated with the edge weights μ_{xy}. The Markov chain is defined by the following one-step transition probability

$$P(x,y) = \mathbb{P}^x(X_1 = y) = \frac{\mu_{xy}}{\mu(x)}.$$

We will assume that there exists $p_0 > 0$ such that

$$P(x,y) \geq p_0 \tag{5}$$

for all x, y such that $x \sim y$. We also consider P as a Markov operator which acts on functions of Γ by

$$Pf(x) = \sum_{y\in\Gamma} P(x,y) f(y).$$

We will denote non-negative integers by $\mathbb{N} = \{0, 1, 2, \ldots\}$ and positive integers by $\mathbb{N}^* = \{1, 2, 3, \ldots\}$. For any non-negative integer n, the n-step transition probability P_n is defined by $P_n(x, y) = \mathbb{P}(X_n = y \mid X_0 = x) = \mathbb{P}^x(X_n = y)$. Define the *heat kernel* of weighted graph (Γ, μ) by

$$p_n(x, y) := \frac{P_n(x, y)}{\mu(y)}.$$

This Markov chain is symmetric with respect to the measure μ, that is $p_n(x, y) = p_n(y, x)$ for all $x, y \in \Gamma$ and for all $n \in \mathbb{N}$. We assume that there exists $\gamma > 1$ such that the following *sub-Gaussian* estimates are true for the heat kernel p_n. There exist constants $c, C > 0$ such that, for all $x, y \in \Gamma$

$$p_n(x, y) \leq \frac{C}{V(n^{1/\gamma})} \exp\left[-\left(\frac{d(x, y)^\gamma}{Cn} \right)^{\frac{1}{\gamma-1}} \right], \forall n \geq 1 \tag{6}$$

and

$$(p_n + p_{n+1})(x, y) \geq \frac{c}{V(n^{1/\gamma})} \exp\left[-\left(\frac{d(x, y)^\gamma}{cn} \right)^{\frac{1}{\gamma-1}} \right], \forall n \geq 1 \vee d(x, y). \tag{7}$$

Let $\langle \cdot, \cdot \rangle$ denote the inner product in $\ell^2(\Gamma, \mu)$. For the Markov operator P, define the corresponding Dirichlet form $\mathcal{E}_P$ by

$$\mathcal{E}_P(f) := \langle (I - P)f, f \rangle = \frac{1}{2} \sum_{x, y \in \Gamma} (f(x) - f(y))^2 \mu_{xy}$$

for all $f \in \ell^2(\Gamma, \mu)$. For any two sets $A, B \subset \Gamma$, the resistance $R_P(A, B)$ is defined by

$$R_P(A, B)^{-1} = \inf \left\{ \mathcal{E}_P(f, f) : f \in \mathbb{R}^\Gamma, f\big|_A \equiv 1, f\big|_B \equiv 0 \right\}$$

where $\inf \emptyset = +\infty$. By [21, Theorem 3.1], we have the following estimate for the resistance. There exist constants $C_R, A > 1$ such that

$$C_R^{-1} \frac{r^\gamma}{V(r)} \leq R_P(B(x, r), B(x, Ar)^c) \leq C_R \frac{r^\gamma}{V(r)} \tag{8}$$

for all $x \in \Gamma$ and for all $r \geq 1$. Other related work that characterizes the sub-Gaussian estimates (6) and (7) are [20] and [3].

The parameter γ in (6) and (7) is sometimes called the '*escape time exponent*' or '*anomalous diffusion exponent*' or '*walk dimension*'. It is known that $\gamma \geq 2$ necessarily (see for instance [13, Theorem 4.6]). For any $\alpha \in [1, \infty)$ and for any $\gamma \in [2, \alpha + 1]$, Barlow constructs graphs of polynomial volume growth satisfying $V(x, r) \simeq (1 + r)^\alpha$ and sub-Gaussian estimates (6) and (7) (see [1, Theorem 2] and [21, Theorem 3.1]). Moreover, these are the complete range of α and γ for which sub-Gaussian estimates with escape rate exponent γ could possibly hold for graphs of polynomial growth with growth exponent α.

Let $\phi : [0, \infty) \to [1, \infty)$ be a continuous, regularly varying function of positive index. We say a Markov operator K satisfies (J_ϕ) if it has symmetric kernel k

with respect to the measure μ and if there exists a constant $C_\phi > 0$ such that

$$C_\phi^{-1} \frac{1}{V(d(x,y))\phi(d(x,y))} \leq k(x,y) = k(y,x) \leq C_\phi \frac{1}{V(d(x,y))\phi(d(x,y))} \qquad (J_\phi)$$

for all $x, y \in \Gamma$. Let $k_n(x,y)$ denote the kernel of the iterated power K^n with respect to the measure μ. If K satisfies (J_ϕ) and if ϕ is regularly varying with index $\beta > 0$, then we say that β is the *jump index* of the random walk driven by K. Here by random walk driven by K, we mean the discrete time Markov chain $(Y_n)_{n\in N}$ with transition probabilities given by

$$\mathbb{P}(Y_1 = y | Y_0 = x) = K\mathbb{1}_y(x) = k(x,y)\mu(y).$$

We demonstrate threshold behavior as the jump index β varies by analyzing the function

$$\psi_K(n) = \left\|K^{2n}\right\|_{1\to\infty} = \|K^n\|_{1\to 2}^2 = \sup_{x\in\Gamma} k_{2n}(x,x) = \sup_{x,y\in\Gamma} k_{2n}(x,y) \qquad (9)$$

as $n \to \infty$ (see [17] for a proof of (9)). The following theorem gives bounds on $\psi_K(n)$ that are sharp up to constants.

Theorem 1.1. *Let (Γ, μ) be a weighted graph satisfying (1), (2), (3), (5) and suppose that its heat kernel p_n satisfies the sub-Gaussian bounds (6) and (7) with escape time exponent γ. Let K be a Markov operator symmetric with respect to the measure μ satisfying (J_ϕ), where $\phi : [0,\infty) \to [1,\infty)$ is a continuous regularly varying function of positive index. Then there exists a constant $C > 0$ such that*

$$\frac{C^{-1}}{V(\zeta(n))} \leq \psi_K(n) \leq \frac{C}{V(\zeta(n))} \qquad (10)$$

for all $n \in \mathbb{N}$, where $\zeta : [0,\infty) \to [1,\infty)$ is a continuous non-decreasing function which is an asymptotic inverse of $t \mapsto t^\gamma / \int_0^t \frac{s^{\gamma-1}\,ds}{\phi(s)}$.

Example. We write ϕ in Theorem 1.1 as $\phi(t) = ((1+t)l(t))^\beta$ where l is a slowly varying function (we refer the reader to [8, Chap. I] for a textbook introduction on slowly and regularly varying functions). The function ζ of Theorem 1.1 can be described more explicitly as follows:

- If $\beta > \gamma$, $\zeta(t) \simeq t^{1/\gamma}$.
- If $\beta < \gamma$, we have $t^\gamma / \int_0^t \frac{s^{\gamma-1}\,ds}{\phi(s)} \simeq \phi(t)$ and ζ is essentially the asymptotic inverse of ϕ, namely
$$\zeta(t) \simeq t^{1/\beta} l_\#(t^{1/\beta})$$
where $l_\#$ is the de Bruijn conjugate of l. For instance, if l has the property that $l(t^a) \simeq l(t)$ for all $a > 0$, then $l_\# \simeq 1/l$.
- If $\beta = \gamma$, the situation is more subtle. The function $\eta(t) = t^\gamma / \int_0^t \frac{s^{\gamma-1}\,ds}{\phi(s)}$ is regularly varying of index γ and $\eta(t) \leq C_1\phi(t)$ for some constant C_1. For example if $l \equiv 1$, we have $\eta(t) \simeq t^\gamma / \log t$ and $\zeta(t) \simeq (t \log t)^{1/\gamma}$. When $l(t) \simeq (\log t)^{\rho/\gamma}$ with $\rho \in \mathbb{R}$, then
  - If $\rho > 1$, $\eta(t) \simeq t^\gamma$ and $\zeta(t) \simeq t^{1/\gamma}$.

– If $\rho = 1$, $\eta(t) \simeq t^{\gamma}/\log\log t$ and $\zeta(t) \simeq (t\log\log t)^{1/\gamma}$.
– If $\rho < 1$, $\eta(t) \simeq t^{\gamma}/(\log t)^{1-\rho}$ and $\zeta(t) \simeq \big(t(\log t)^{1-\rho}\big)^{1/\gamma}$.

Remark.

(a) The conclusion of Theorem 1.1 holds if K is symmetric with respect to a different measure ν that is comparable to the counting measure in the sense described by (1). This can be seen by comparing ψ_K with ψ_{Q_ϕ} where Q_ϕ will be defined in (29). We simply use the definition (9) along with the fact that $L^p(\Gamma,\nu)$ and $L^p(\Gamma,\mu)$ have comparable norms.

(b) The condition (5) is required only for the lower bound on ψ_K.

(c) Let ϕ in Theorem 1.1 be regularly varying with index $\beta > 0$. If $\beta \in (0,2)$ we know matching two sided estimates on $k_n(x,y)$ for all $n \in \mathbb{N}$ and for all $x,y \in \Gamma$. Assume that $\phi(t) = ((1+t)l(t))^{\beta}$ where l is a slowly varying function. The main result of [15] states that

$$k_n(x,y) \simeq \left(\frac{1}{V(n^{1/\beta}l_{\#}(n^{1/\beta}))} \wedge \frac{n}{V(d(x,y))\phi(d(x,y))}\right), \tag{11}$$

where $l_{\#}$ is the de Bruijn conjugate of l.

(d) We conjecture that the two-sided estimate (11) is true for any $\beta \in (0,\gamma)$, where γ is the escape time exponent for the sub-Gaussian estimate in (6) and (7).lem The proof of (11) in [15] doesn't seem to work if $\beta \in [2,\gamma)$. In particular, the use of Davies' method to prove off-diagonal upper bounds does not seem to work directly.

(e) The conclusion of Theorem 1.1 can be strengthened for random walks on groups for all values of β (γ is necessarily 2 for random walks on groups). See [24, Theorem 1.5] for more.

(f) Another intriguing question is to find matching two-sided estimates $k_n(x,y)$ for the case $\beta \geq \gamma$ for appropriate range of $d(x,y)$. In light of [24, Theorem 1.5] for random walks on groups, we conjecture that

$$k_n(x,y) \simeq \frac{1}{V(\zeta(n))}$$

for all $n \in \mathbb{N}^*$ and for all $x,y \in \Gamma$ such that $d(x,y) \leq \zeta(n)$.

(g) It is a technically challenging open problem to replace the homogeneous volume doubling assumptions (2) and (3) by the more general volume doubling assumption: there exists $C_D > 0$ such that $V(x,2r) \leq C_D V(x,r)$ for all $x \in \Gamma$ and for all $r > 0$.

Theorem 1.1 indicates a possible moment threshold behavior. We define moment of random walk as follows.

Definition 1.2. For a Markov operator K on Γ and any number $r > 0$, we define the r-moment of random walk driven by K as

$$M_{r,K} := \sup_{x\in\Gamma} \mathbb{E}^x d(X_0,X_1)^r = \sup_{x\in\Gamma} \left(K(d_x^r)\right)(x)$$

where $(X_n)_{n\in\mathbb{N}}$ is a random walk driven by the Markov operator K and $d_x^r : \Gamma \to \mathbb{R}$ denotes the function $y \mapsto (d(x,y))^r$.

Here is a corollary of Theorem 1.1 that illustrates moment threshold behavior of random walks. It states that the asymptotic behavior of ψ_K is same as ψ_P corresponding to the natural random walk if and only if K has finite γ-moment.

Corollary 1.3. *Let (Γ, μ) be an infinite, weighted graph satisfying (1), (2), (3), (5) and its heat kernel p_n satisfies the sub-Gaussian bounds (6) and (7) with escape time exponent γ. Let K be a Markov operator symmetric with respect to the measure μ satisfying (J_ϕ), where $\phi : [0,\infty) \to [1,\infty)$ is a continuous regularly varying function of positive index. Then the following are equivalent:*

(a) K has finite γ-moment, that is $M_{\gamma,K} < \infty$.
(b) There exists a constant $C > 0$ such that

$$\frac{C^{-1}}{V(n^{1/\gamma})} \le \psi_K(n) \le \frac{C}{V(n^{1/\gamma})} \tag{12}$$

for all $n \in \mathbb{N}$.

Remark. For random walks on groups one must have $\gamma = 2$ and such a second moment threshold behavior is known in greater generality [16, Theorem 1.4 and Corollary 1.5]. See [6], [7] and [24] for extensions and generalizations of such moment threshold behavior for random walks on groups. It is an interesting open problem to formulate and prove a γ-moment threshold in greater generality without the assumption (J_ϕ).

Proof of Corollary 1.3. By Theorem 1.1, (b) holds if and only if $\int_0^\infty \frac{s^{\gamma-1}\,ds}{\phi(s)} < \infty$. Therefore (b) holds if and only if

$$\sum_{n=1}^{\infty} \frac{n^{\gamma-1}}{\tilde{\phi}(n)} < \infty, \tag{13}$$

where $\tilde{\phi}(x) = \sup_{t\in[0,x]} \phi(x)$. The above statement follows from Potter's bounds [8, Theorem 1.5.6], continuity of ϕ, Theorem 1.5.3 of [8] and uniqueness of asymptotic inverse up to asymptotic equivalence.

By (J_ϕ) and Theorem 1.5.3 of [8], the condition $M_{\gamma,K} < \infty$ holds if and only if

$$\sum_{y\in\Gamma} \frac{d(x,y)^\gamma}{V(d(x,y))\phi(d(x,y))} < \infty \tag{14}$$

for some fixed $x \in \Gamma$. It is well-known that the volume doubling property (2) and (3) implies a reverse volume doubling property which has the following consequence: There exists an integer $A \in \mathbb{N}^*$ and $c_1 > 0$ such that

$$V(x, Ar) - V(x,r) \ge c_1 V(r) \tag{15}$$

for all $r \geq 1/2$ (Proof of [19, Proposition 3.3] goes through with minor modifications). There exists $c_2, c_3 > 0$ such that

$$\begin{aligned}
&\sum_{y\in\Gamma} \frac{d(x,y)^\gamma}{V(d(x,y))\phi(d(x,y))} \\
&\geq c_2 \sum_{n=0}^{\infty} \sum_{y\in B(x,A^{n+1}/2)\setminus B(x,A^n/2)} \frac{A^{n\gamma}}{V(A^{n+1}/2)\tilde{\phi}(A^{n+1}/2)} \\
&\geq c_3 \sum_{n=0}^{\infty} \frac{A^{n\gamma}}{\tilde{\phi}(A^n)} \qquad (16)
\end{aligned}$$

for all $x \in \Gamma$.

Now we show a reverse inequality of (16).There exists $C_1, C_2 > 0$ such that

$$\begin{aligned}
\sum_{y\in\Gamma} \frac{d(x,y)^\gamma}{V(d(x,y))\phi(d(x,y))} &\leq C_1 \sum_{n=0}^{\infty} \sum_{y\in B(x,2^n)\setminus B(x,2^{n-1})} \frac{2^{n\gamma}}{V(2^{n-1})\tilde{\phi}(2^{n-1})} \\
&\leq C_2 \sum_{n=0}^{\infty} \frac{2^{n\gamma}}{\tilde{\phi}(2^n)} \qquad (17)
\end{aligned}$$

for all $x \in \Gamma$. The second line above follows from (2) and Potter's bound [8, Theorem 1.5.6].

To show (a) implies (b), we use (14), (16) and a generalization of Cauchy condensation test due to Schlömilch to obtain (13). To show (b) implies (a), we use (13), Cauchy condensation test and (17) to obtain (14) which implies (b). □

Theorem 1.1 and Corollary 1.3 suggests that for spaces with sub-Gaussian estimates and a scaling structure (for example regular fractals), one might be able to formulate and prove a central limit theorem with a $\gamma + \epsilon$ moment condition.

1.1. **Analytic preliminaries on Markov operator and Dirichlet form.** Let (Γ, μ) be a countable, weighted graph. Let K be a Markov operator, symmetric with respect to the measure μ. Denote the kernel of the iterated operator K^n with respect to μ by $k_n(x,y)$, that is $K^n f(x) = \sum_{y\in\Gamma} k_n(x,y) f(y)\mu(y)$. We will collect some useful facts about the operator K. For any $p \in [1,\infty]$, we denote by $\|f\|_p$ the norm of f in $\ell^p(\Gamma,\mu)$ and by $\langle\cdot,\cdot\rangle$ the inner product in $\ell^2(\Gamma,\mu)$. A fundamental property of K is that it is a contraction in $\ell^p(\Gamma)$ for any $p \in [1,\infty]$, that is

$$\|Kf\|_p \leq \|f\|_p$$

for all $p \in [1,\infty]$ and for all $f \in \ell^p(\Gamma,\mu)$. By the symmetry $k_1(x,y) = k_1(y,x)$, we have that K is self-adjoint in $\ell^2(\Gamma,\mu)$, that is

$$\langle Kf, g\rangle = \langle f, Kg\rangle \qquad (18)$$

for all $f, g \in \ell^2(\Gamma,\mu)$. For any $n \in \mathbb{N}$, we denote by $\mathcal{E}_{K^n}(f,f) = \langle (I-K^n)f, f\rangle$ the Dirichlet form associated with K^n.

The following useful lemma compares Dirichlet form of a Markov operator K with its iterated power K^n.

Lemma 1.4 (Folklore)**.** *Let K be a Markov operator on Γ symmetric with respect to the measure μ. Then for any $f \in \ell^2(\Gamma,\mu)$ and for any $n \in \mathbb{N}^*$*

$$\mathcal{E}_{K^n}(f,f) \le n\mathcal{E}_K(f,f). \tag{19}$$

Proof. We verify this using spectral theory. Let E_λ be the spectral resolution of K. Therefore

$$\mathcal{E}_{K^n}(f,f) - n\mathcal{E}_K(f,f) = \int_{-1}^{1} (1-\lambda^n - n + n\lambda) dE_\lambda(f,f).$$

The result follows from the observation that $1-\lambda^n - n + n\lambda \le 0$ for all $\lambda \in [-1,1]$ and for all $n \in \mathbb{N}^*$. □

Lemma 1.5 (Folklore)**.** *Let K be a Markov operator on Γ symmetric with respect to the measure μ and let $f \in \ell^2(\Gamma,\mu)$ be a non-zero function. Then the function $i \mapsto \left\|K^i f\right\|_2 / \left\|K^{i-1} f\right\|_2$ is non-decreasing.*

Proof. We use self-adjointness (18) and Cauchy-Schwarz inequality to obtain

$$\left\|K^i f\right\|_2^2 = \langle K^{i-1}f, K^{i+1}f\rangle \le \left\|K^{i-1} f\right\|_2 \left\|K^{i+1} f\right\|_2,$$

which gives the desired result. □

2. Pseudo-Poincaré inequality using Discrete subordination

Pseudo-Poincaré inequality provides an efficient way to prove Nash inequality which in turn gives upper bounds on $\psi_K(n)$. For a function $f : \Gamma \to \mathbb{R}$ and $R > 0$, we define a function $f_R : \Gamma \to \mathbb{R}$ by

$$f_R(x) := \frac{1}{V(x,R)} \sum_{y \in B(x,R)} f(y)\mu(y).$$

In other words, $f_R(x)$ is the μ-average of f in $B(x,R)$. The main result of the section is the following pseudo-Poincaré inequality.

Proposition 2.1 (Pseudo-Poincaré inequality)**.** *Let (Γ,μ) be a weighted graph satisfying (1), (2), (3) and suppose that its heat kernel p_n satisfies the sub-Gaussian bounds (6) and (7) with escape time exponent γ. Let K be a Markov operator symmetric with respect to the measure μ satisfying (J_ϕ), where $\phi : [0,\infty) \to [1,\infty)$ is a continuous regularly varying function of positive index. There exists a constant $C > 0$ such that*

$$\|f - f_R\|_2^2 \le C \left(\frac{R^\gamma}{\int_0^R \frac{s^{\gamma-1}\, ds}{\phi(s)}} \right) \mathcal{E}_K(f,f) \tag{20}$$

for all $R > 0$ and for all $f \in \ell^2(\Gamma,\mu)$.

We introduce a discrete subordination of the natural random walk on (Γ,μ) whose kernel is comparable to the kernel of K in Proposition 2.5. We introduce a new Markov operator

$$Q := \frac{1}{2}(P + P^2) \tag{21}$$

which has a symmetric kernel $q(x,y) = \frac{1}{2}(p_1(x,y) + p_2(x,y))$ with respect to μ. Let q_k denote the kernel of the Markov operator Q^k. For a Markov operator Q^k, let $\mathcal{E}_{Q^k}(f,f) := \langle (I - Q^k)f, f\rangle$ denote the corresponding Dirichlet form. Let R_Q denote the resistance defined using the Dirichlet form $\mathcal{E}_Q$. We will now compare kernels of P^k and Q^k.

Remark. The advantage of working with the kernel q_n is that it satisfies as stronger sub-Gaussian lower estimate (23) in comparison to (7) satisfied by p_n. This makes subordination of kernel Q preferable(as opposed to P) and technically easier.

Lemma 2.2. *The kernel q_k satisfies the following improved sub-Gaussian estimates: there exist constants $c, C > 0$ such that, for all $x, y \in \Gamma$*

$$q_n(x,y) \leq \frac{C}{V(n^{1/\gamma})} \exp\left[-\left(\frac{d(x,y)^\gamma}{Cn}\right)^{\frac{1}{\gamma-1}}\right], \forall n \geq 1 \tag{22}$$

and

$$q_n(x,y) \geq \frac{c}{V(n^{1/\gamma})} \exp\left[-\left(\frac{d(x,y)^\gamma}{cn}\right)^{\frac{1}{\gamma-1}}\right], \forall n \geq 1 \vee d(x,y). \tag{23}$$

Proof. Observe that $q_n(x,y) = \sum_{k=0}^n 2^{-n}\binom{n}{k} p_{n+k}(x,y)$. This along with (6), (2) gives the desired upper bound (22).

Note that, there exists $C_1 > 1$ such that

$$C_1^{-1} \leq \frac{\binom{n}{k}}{\binom{n}{k+1}} \leq C_1 \tag{24}$$

for all $n \in \mathbb{N}^*$ and for all $k \in \mathbb{N}$ such that $\lfloor \frac{n}{4} \rfloor \leq k \leq \lfloor \frac{3n}{4} \rfloor$. There exists $c_1, c_2 > 0$

$$\begin{aligned}
q_n(x,y) &\geq \sum_{k=\lfloor \frac{n}{4} \rfloor}^{\lfloor \frac{3n}{4} \rfloor + 1} 2^{-n} \binom{n}{k} p_{n+k}(x,y) \\
&\geq 2^{-n-1} C_1^{-1} \sum_{k=\lfloor \frac{n}{4} \rfloor}^{\lfloor \frac{3n}{4} \rfloor} \binom{n}{k} (p_{n+k}(x,y) + p_{n+k+1}(x,y)) \\
&\geq 2^{-n-1} c_1 C_1^{-1} C_D^{-1} \frac{1}{V(n^{1/\gamma})} \exp\left[-\left(\frac{d(x,y)^\gamma}{c_1 n}\right)^{\frac{1}{\gamma-1}}\right] \sum_{k=\lfloor \frac{n}{4} \rfloor}^{\lfloor \frac{3n}{4} \rfloor} \binom{n}{k} \\
&\geq c_2 \frac{1}{V(n^{1/\gamma})} \exp\left[-\left(\frac{d(x,y)^\gamma}{c_1 n}\right)^{\frac{1}{\gamma-1}}\right]
\end{aligned}$$

for all $x, y \in \Gamma$ and for all $n \in \mathbb{N}$ such that $n \geq 1 \vee d(x,y)$. The second line above follows from (24), the third line follows from (7) and (2) and the last line follows from weak law of large numbers. □

The operators P and Q have comparable Dirichlet forms and resistances.

Lemma 2.3. *The resistances R_Q and R_P are comparable by the following inequality*

$$\frac{1}{2}R_P(f,f) \le R_Q(A,B) \le 2R_P(A,B)$$

for all subsets $A, B \subset \Gamma$.

Proof. It suffices to compare the corresponding Dirichlet forms $\mathcal{E}_Q$ and $\mathcal{E}_P$. Note that $\mathcal{E}_Q(f,f) = \frac{1}{2}(\mathcal{E}_P(f,f) + \mathcal{E}_{P^2}(f,f)) \ge \frac{1}{2}\mathcal{E}_P(f,f)$. However by Lemma 1.4, we have

$$\mathcal{E}_Q(f,f) = \frac{1}{2}(\mathcal{E}_P(f,f) + \mathcal{E}_{P^2}(f,f)) \le \frac{3}{2}\mathcal{E}_P(f,f) \le 2\mathcal{E}_P(f,f).$$

□

We have the following pseudo-Poincaré inequality for iterated powers of Q.

Lemma 2.4. *Under the assumptions of Proposition 2.1, there exists $C_1 > 0$ such that*

$$\|f - f_R\|_2^2 \le C_1 \left(\frac{R}{k}\right)^\gamma \mathcal{E}_{Q^{2\lfloor k^\gamma \rfloor}}(f,f)$$

for all $f \in \ell^2(\Gamma,\mu)$ and for all $k \in \mathbb{N}$ and $R \in \mathbb{R}$ satisfying $1 \le k \le R$.

Proof. There exists $C_2 > 0$ such that

$$\begin{aligned}
\|f - f_R\|^2 &\le \sum_{x\in\Gamma}\sum_{y\in B(x,R)} \frac{(f(x)-f(y))^2}{V(x,R)}\mu(y)\mu(x) \\
&\le C_2 \sum_{x\in\Gamma}\sum_{y\in\Gamma}(f(x)-f(y))^2 q_{2\lfloor R^\gamma\rfloor}(x,y)\mu(y)\mu(x) \\
&= 2C_2\left(\|f\|_2^2 - \left\|Q^{\lfloor R^\gamma\rfloor} f\right\|_2^2\right). \qquad (25)
\end{aligned}$$

The first line follows from Jensen's inequality, the second line follows from the lower bound (23) of Lemma 2.2, (4) and (3), the last line follows from the μ-symmetry of Q. Since Q is a contraction on $\ell^2(\Gamma,\mu)$, we have

$$\|f\|_2^2 - \left\|Q^{\lfloor R^\gamma\rfloor}f\right\|_2^2 \le \|f\|_2^2 - \left\|Q^{l\lfloor k^\gamma\rfloor}f\right\|_2^2 = \sum_{m=0}^{l-1}\left(\left\|Q^{m\lfloor k^\gamma\rfloor}f\right\|_2^2 - \left\|Q^{(m+1)\lfloor k^\gamma\rfloor}f\right\|_2^2\right) \qquad (26)$$

where $l = \lceil \lfloor R^\gamma \rfloor / \lfloor k^\gamma \rfloor \rceil$.

Since Q is a contraction on $\ell^2(\Gamma,\mu)$, we have

$$\begin{aligned}
\left\|Q^{(m+1)\lfloor k^\gamma\rfloor}f\right\|_2^2 - \left\|Q^{(m+2)\lfloor k^\gamma\rfloor}f\right\|_2^2 &= \left\|Q^{(m+1)\lfloor k^\gamma\rfloor}(I - Q^{2\lfloor k^\gamma\rfloor})^{1/2}f\right\|_2^2 \\
&\le \left\|Q^{m\lfloor k^\gamma\rfloor}(I - Q^{2\lfloor k^\gamma\rfloor})^{1/2}f\right\|_2^2 \\
&= \left\|Q^{m\lfloor k^\gamma\rfloor}f\right\|_2^2 - \left\|Q^{(m+1)\lfloor k^\gamma\rfloor}f\right\|_2^2. \qquad (27)
\end{aligned}$$

By (26) and (27), we get

$$\|f\|_2^2 - \left\|Q^{\lfloor R^\gamma \rfloor} f\right\|_2^2 \le l \left(\|f\|_2^2 - \left\|Q^{\lfloor k^\gamma \rfloor} f\right\|_2^2 \right) \le 4 \frac{R^\gamma}{k^\gamma} \left(\|f\|_2^2 - \left\|Q^{\lfloor k^\gamma \rfloor} f\right\|_2^2 \right). \tag{28}$$

Combining (25) and (28) gives the desired inequality. □

The following subordinated kernel satisfying (J_ϕ) is a useful tool to study the behavior of long range random walks.

Proposition 2.5. *Let $\phi : [0, \infty) \to [1, \infty)$ be a continuous regularly varying function of positive index. Let (Γ, μ) be a weighted graph satisfying the assumptions of Proposition 2.1 and let Q be defined by (21). Define the subordinated Markov kernel*

$$Q_\phi := \sum_{n=1}^{\infty} c_\phi \frac{1}{n\phi(n)} Q^{2\lfloor n^\gamma \rfloor} \tag{29}$$

where $c_\phi = \left(\sum_{n=1}^{\infty} \frac{1}{n\phi(n)} \right)^{-1}$. Then Q_ϕ has a symmetric kernel q_ϕ with respect to μ and there exists $C > 0$ such that

$$C^{-1} \frac{1}{V(d(x,y))\phi(d(x,y))} \le q_\phi(x,y) = q_\phi(y,x) \le C \frac{1}{V(d(x,y))\phi(d(x,y))}. \tag{30}$$

for all $x, y \in \Gamma$. In other words, Q_ϕ satisfies (J_ϕ).

Proof. The symmetry of Q_ϕ follows from the symmetry of Q since

$$q_\phi(x,y) := \sum_{n=1}^{\infty} c_\phi \frac{1}{n\phi(n)} q_{2\lfloor n^\gamma \rfloor}(x,y). \tag{31}$$

Let ϕ be regularly varying of index $\beta > 0$. By Potter's bounds [8, Theorem 1.5.6] and using that ϕ is a positive continuous function, there exists $C_1 > 0$ such that

$$\frac{\phi(s)}{\phi(t)} \le C_1 \max \left(\left(\frac{s}{t}\right)^{3\beta/2}, \left(\frac{s}{t}\right)^{\beta/2} \right) \tag{32}$$

for all $s, t \in [1, \infty)$.

It suffices to assume that $x, y \in \Gamma$ and $x \neq y$. The case $x = y$ follows trivially from Lemma 2.2. Combining $n^\gamma/2 \le \lfloor n^\gamma \rfloor \le n^\gamma$, (31), (2) and (22) of Lemma 2.2, there exists $C_2 > 0$ such that

$$q_\phi(x,y) \le \sum_{n=d(x,y)+1}^{\infty} \frac{C_2}{n\phi(n)V(n)} + \sum_{n=1}^{d(x,y)} \frac{C_2}{n\phi(n)V(n)} \exp\left[-\left(\frac{d(x,y)}{C_2 n} \right)^{\gamma/(\gamma-1)} \right] \tag{33}$$

for all $x, y \in \Gamma$ with $x \neq y$. We bound the first term in (33) by

$$\begin{aligned}\sum_{n=d(x,y)+1}^{\infty} \frac{1}{n\phi(n)V(n)} &\le \frac{1}{V(d(x,y))} \sum_{n=d(x,y)+1}^{\infty} \frac{1}{n\phi(n)} \\ &\le C_3 \frac{1}{V(d(x,y))} \int_{d(x,y)}^{\infty} \frac{ds}{s\phi(s)} \\ &\le C_4 \frac{1}{V(d(x,y))\phi(d(x,y))} \end{aligned} \tag{34}$$

where $C_3, C_4 > 0$ are constants. In the first line above, we used that V is non-decreasing. The second line above follows from (32) and the third line follows from [8, Proposition 1.5.10].

Let $1 \le n \le d(x,y)$. To estimate second term in (33), we use (32) and (4) to obtain

$$\begin{aligned}\frac{1}{n\phi(n)V(n)} &= \frac{1}{d(x,y)\phi(d(x,y))V(d(x,y))} \frac{d(x,y)\phi(d(x,y))V(d(x,y))}{n\phi(n)V(n)} \\ &\le \frac{C_1 C_D}{d(x,y)\phi(d(x,y))V(d(x,y))} \left(\frac{d(x,y)}{n}\right)^{\alpha+((3\beta)/2)+1}. \end{aligned} \tag{35}$$

Since the function $t \mapsto t^{\alpha+((3\beta)/2)+1} \exp\left[-(C_2^{-1}t)^{\gamma/(\gamma-1)}\right]$ is uniformly bounded (by say C_5) in $[1,\infty)$, by (35), there exists a constant $C_6 > 0$ such that

$$\sum_{n=1}^{d(x,y)} \frac{C_2}{n\phi(n)V(n)} \exp\left[-\left(\frac{d(x,y)}{C_2 n}\right)^{\gamma/(\gamma-1)}\right] \le \frac{C_6}{V(d(x,y))\phi(d(x,y))} \tag{36}$$

for all $x, y \in \Gamma$ with $x \neq y$. Combining (33), (34) and (36) gives the desired upper bound in (30).

For the lower bound in (30), we use (31), (23) of Lemma 2.2 along with (4) to obtain, a constant $c_1 > 0$ such that

$$\begin{aligned} q_\phi(x,y) &\ge \sum_{n=d(x,y)}^{2d(x,y)} \frac{c_\phi}{n\phi(n)} q_{2\lfloor n^\gamma \rfloor}(x,y) \\ &\ge \sum_{n=d(x,y)}^{2d(x,y)} \frac{c_1}{n\phi(n)V(n)} \\ &\ge \frac{C_D^{-1} c_1}{2d(x,y)V(d(x,y))\phi(d(x,y))} \sum_{n=d(x,y)}^{2d(x,y)} \frac{1}{3C_1} \end{aligned}$$

for all $x, y \in \Gamma$ with $x \neq y$. In the last line, we used, (2), $n^{-1} \ge (2d(x,y))^{-1}$ and the Potter's bound (32). □

Proof of Proposition 2.1. By Proposition 2.5, the Markov operators K and Q_ϕ have comparable Dirichlet forms. Hence it suffices to consider the case $K = Q_\phi$. If $R < 1$, then $f \equiv f_R$ which in turn implies the pseudo-Poincaré inequality (20).

Hence we assume that $R \geq 1$. There exists $c_1 > 0$ such that

$$\begin{aligned}\mathcal{E}_{Q_\phi}(f,f) &= c_\phi \sum_{k=1}^{\infty} \frac{1}{k\phi(k)} \mathcal{E}_{Q^{2\lfloor k^\gamma \rfloor}}(f,f) \\ &\geq c_\phi C_1^{-1} \|f - f_R\|_2^2 R^{-\gamma} \sum_{k=1}^{\lfloor R \rfloor} \frac{k^{\gamma-1}}{\phi(k)} \\ &\geq c_1 \|f - f_R\|_2^2 R^{-\gamma} \int_0^R \frac{s^{\gamma-1}\,ds}{\phi(s)} \end{aligned} \tag{37}$$

for all $f \in \ell^2(\Gamma,\mu)$ and for all $R > 0$ which is the desired inequality. In the second line above, we used Lemma 2.4 and in the last line we used that ϕ is a positive continuous regularly varying function which satisfies the Potter's bound (32). $\square$

3. Nash inequality and Ultracontractivity.

In this section, we use pseudo-Poincaré inequality (20) to obtain a Nash inequality and on-diagonal upper bounds. A polished treatment of the relationship between Nash inequalities and ultracontractivity is presented in [9]. It is well-known that pseudo-Poincaré inequality along with assumptions on volume growth gives a Sobolev-type inequality (see [23, Theorem 2.1] for an early reference to this approach).

The following function η which appears in (20) plays a crucial role in this work. Define the function $\eta : [0,\infty) \to (0,\infty)$

$$\eta(R) := \frac{R^\gamma}{\int_0^R \frac{s^{\gamma-1}\,ds}{\phi(s)}} \tag{38}$$

for $R > 0$ and $\eta(0) = \gamma\phi(0)$ so that η is a continuous function. We also need the following modification of η defined as $\tilde{\eta} : [0,\infty) \to (0,\infty)$

$$\tilde{\eta}(R) := \sup\{\eta(t) : t \in [0,R]\} \tag{39}$$

so that $\tilde{\eta}$ is a non-decreasing function. It is known that [8, Theorem 1.5.3] $\tilde{\eta}$ is asymptotically equivalent to η, that is $\lim_{t\to\infty} \tilde{\eta}(t)/\eta(t) = 1$. If ϕ is regularly varying with positive index, so is η. We now compute the index of η and list some of its basic properties.

Lemma 3.1. *If $\phi : [0,\infty) \to [1,\infty)$ is a continuous regularly varying function with index $\beta > 0$, then*

(a) The function η defined by (38) is continuous, positive and regularly varying with index $\beta \wedge \gamma$.
(b) There exists $C_1 > 0$ such that $\eta(x) \leq C_1\phi(x)$ for all $x \geq 0$.
(c) The function η has an asymptotic inverse $\zeta : [0,\infty) \to [1,\infty)$ satisfying the following properties: ζ is continuous, non-decreasing and regularly varying with index $1/(\beta \wedge \gamma)$. Moreover, there exists $C > 0$ such that

$$C^{-1}t \leq \zeta(\eta(t)) \leq \zeta(\tilde{\eta}(t)) \leq Ct \qquad \textit{and} \qquad C^{-1}t \leq \eta(\zeta(t)) \leq \tilde{\eta}(\zeta(t)) \leq Ct \tag{40}$$

for all $t \geq 1$.

Proof. (a) and (b): The cases $\beta < \gamma$, $\beta = \gamma$ and $\beta > \gamma$ follow from Proposition 1.5.8, Proposition 1.5.9a and Proposition 1.5.10 in [8] respectively.

(c) The existence of an asymptotic inverse which is regularly varying of index $1/(\beta \wedge \gamma)$ follows from (a) and [8, Proposition 1.5.12]. The fact that ζ can be chosen to be continuous, bounded below by 1 and non-decreasing follows from Theorem 1.8.2, Proposition 1.5.1 and Theorem 1.5.3 of [8] respectively. The existence of $C > 0$ satisfying (40) follows from the definition of asymptotic inverse and continuity of ζ,η and $\tilde{\eta}$ and $\lim_{t\to\infty} \tilde{\eta}(t)/\eta(t) = 1$. ◻

Theorem 3.2 (Nash inequality)**.** *Let* $\phi : [0,\infty) \to [1,\infty)$ *be a continuous, regularly varying function of positive index. Let* K *be Markov operator satisfying* (J_ϕ) *with symmetric kernel* k *with respect to the measure* μ*. Then there exist constants* $C_1, C_2 > 0$ *such that*

$$\|f\|_2^2 \leq C_1 \mathcal{E}_{K^2}(f,f) \tilde{\eta}\left(V^{-1}\left(C_2 \frac{\|f\|_1^2}{\|f\|_2^2} \right)\right) \tag{41}$$

for all $f \in \ell^1(\Gamma,\mu)$*, where* $\tilde{\eta}$ *is given by* (38) *and* (39)*.*

Proof. Let $R > 0$ and $f \in \ell^1(\Gamma,\mu)$.

By (3) and triangle inequality, we have

$$\|f_R\|_\infty \leq C_h \|f\|_1 / V(R) \qquad \text{and} \qquad \|f_R\|_1 \leq C_h^2 \|f\|_1 .$$

Hence by Hölder's inequality

$$\|f_R\|_2^2 \leq \|f_R\|_\infty \|f_R\|_1 \leq C_h^3 \frac{\|f\|_1^2}{V(R)} \tag{42}$$

for all $f \in \ell^1(\Gamma,\mu)$ and for all $R > 0$. By (42) and Proposition 2.1, there exists $C_3 > 0$ such that

$$\begin{aligned} \|f\|_2^2 &\leq 2\|f - f_R\|_2^2 + 2\|f_R\|_2^2 \\ &\leq C_3 \left(\eta(R)\mathcal{E}_K(f,f) + \frac{\|f\|_1^2}{V(R)} \right) \\ &\leq C_3 \left(\tilde{\eta}(R)\mathcal{E}_K(f,f) + \frac{\|f\|_1^2}{V(R)} \right). \end{aligned} \tag{43}$$

To minimize (43), we want to choose $R = R_0 > 0$ such that $(\tilde{\eta}(R_0)V(R_0))^{-1} \simeq \mathcal{E}_K(f,f)/\|f\|_1^2$.

Note that $R \mapsto (\tilde{\eta}(R)V(R))^{-1}$ is a strictly decreasing continuous function with

$$\lim_{R\to 0^+} (\tilde{\eta}(R)V(R))^{-1} = (\eta(0)V(0))^{-1} \qquad \text{and} \qquad \lim_{R\to\infty} (\tilde{\eta}(R)V(R))^{-1} = 0.$$

Therefore the equation

$$(\tilde{\eta}(R)V(R))^{-1} = t \tag{44}$$

has an unique solution for all $t \in \left(0, (\eta(0)V(0))^{-1}\right]$.

Since K is a contraction in $\ell^2(\Gamma,\mu)$, we have

$$\mathcal{E}_K(f,f) = \langle (I-K)f,f\rangle \le \|f\|_2^2 + |\langle Kf,f\rangle| \le \|f\|_2^2 + \|f\|_2 \|Kf\|_2 \le 2\|f\|_2^2 .$$

By (1) and using ℓ^p inequalities for counting measure, we have $\|f\|_1^2 \ge C_\mu^{-3}\|f\|_2^2$. Combining these observations gives

$$\mathcal{E}_K(f,f)/\|f\|_1^2 \le 2C_\mu^3 \tag{45}$$

for all $f \in \ell^1(\Gamma,\mu)$. By (44) and (45), for any $f \in \ell^1(\Gamma,\mu)$ with $f \neq 0$, there exists an unique solution R_0 to the equation

$$(\tilde\eta(R_0)V(R_0))^{-1} = c_1 \frac{\mathcal{E}_K(f,f)}{\|f\|_1^2}, \tag{46}$$

where $c_1 = \left(2C_\mu^3\eta(0)V(0)\right)^{-1}$. Substituting the above solution R_0 in (43) gives $\|f\|_2^2 \le C_3(1+c_1^{-1})\|f\|_1^2/V(R_0)$ or equivalently,

$$R_0 \le V^{-1}\left(C_2\|f\|_1^2/\|f\|_2^2\right) \tag{47}$$

where $C_2 := C_3(1+c_1^{-1})$. Since $\tilde\eta$ is a non-decreasing function, by (46) and (47) we have

$$\frac{\|f\|_1^2}{\mathcal{E}_K(f,f)} \le c_1C_2\frac{\|f\|_1^2}{\|f\|_2^2}\tilde\eta\left(V^{-1}\left(C_2\frac{\|f\|_1^2}{\|f\|_2^2}\right)\right).$$

Hence we obtain the Nash inequality

$$\|f\|_2^2 \le c_1C_2\mathcal{E}_K(f,f)\tilde\eta\left(V^{-1}\left(C_2\frac{\|f\|_1^2}{\|f\|_2^2}\right)\right). \tag{48}$$

By (J_ϕ) and (1), there exists $\alpha > 0$ such that $\inf_{x\in\Gamma} k_1(x,x)\mu(x) \ge \alpha$. Since $k_2(x,y) \ge k_1(x,y)k_1(y,y)\mu(y) \ge \alpha k_1(x,y)$, we have

$$\mathcal{E}_K(f,f) \le \alpha^{-1}\mathcal{E}_{K^2}(f,f)$$

for all $f \in \ell^2(\Gamma,\mu)$. This along with (48) gives the desired Nash inequality. □

Theorem 3.3 (Ultracontractivity). *Let (Γ,μ) be a weighted graph satisfying (1), (2), (3) and its heat kernel p_n satisfies the sub-Gaussian bounds (6) and (7) with escape time exponent γ. Let K be a Markov operator symmetric with respect to the measure μ satisfying (J_ϕ), where $\phi : [0,\infty) \to [1,\infty)$ is a continuous regularly varying function of positive index. Then there exists a constant $C > 0$ such that*

$$\psi_K(n) \le \frac{C}{V(\zeta(n))}$$

for all $n \in \mathbb{N}$, where $\zeta : [0,\infty) \to [1,\infty)$ is a continuous non-decreasing function which is an asymptotic inverse of $t \mapsto t^\gamma / \int_0^t \frac{s^{\gamma-1}\,ds}{\phi(s)}$.

Proof. Let $\mu_* = \inf_{x\in\Gamma}\mu(x)$. Define $g : (0, 1/\mu_*] \to [0,\infty)$ by

$$g(t) := \int_{\mu_*}^{1/t} C_1\tilde\eta\left(V^{-1}(C_2 s)\right)\frac{ds}{s}$$

and $m : [0,\infty) \to (0, 1/\mu_*]$ as the inverse of g, where C_1, C_2 are constants from (41). Since g is a decreasing, surjective, continuous function, so is m. Observe that we can increase the constant C_2 in (41) without affecting the Nash inequality. We choose C_2 such that $C_2 \ge V(1)/\mu_*$, so that

$$V^{-1}(C_2 s) \ge 1 \tag{49}$$

for all $s \ge \mu_*$.

By a standard ultracontractivity estimate using Nash inequality (41) (see [17, Theorem 2.2.1] or [9, Proposition IV.1]), we obtain

$$\psi_K(n) \le m(n) \tag{50}$$

for all $n \in \mathbb{N}^*$.

We now estimate the functions $g(t)$ and its inverse $m(t)$. For $t^{-1} \ge \mu_*$, choose $L \in \mathbb{N}$ such that $C_D^L \mu_* \in [t^{-1}, C_D t^{-1})$. We have

$$\begin{aligned}
g(t) &\le \int_{\mu_*}^{C_D^L \mu_*} C_1 \tilde{\eta}\left(V^{-1}(C_2 s)\right) \frac{ds}{s} = C_1 \int_{C_2\mu_*}^{C_D^L C_2 \mu_*} \tilde{\eta}\left(V^{-1}(s)\right) \frac{ds}{s} \\
&\le C_1 \sum_{k=1}^{L} \int_{C_D^{k-1} C_2 \mu_*}^{C_D^k C_2 \mu_*} \frac{\tilde{\eta}(V^{-1}(s))\, ds}{s} \\
&\le C_1 \sum_{k=1}^{L} \frac{\tilde{\eta}(V^{-1}(C_D^k C_2 \mu_*))}{C_D^{k-1} C_2 \mu_*} \left((C_D - 1) C_D^{k-1} C_2 \mu_*\right) \\
&\le C_3 \sum_{k=1}^{L} \tilde{\eta}(V^{-1}(C_D^k C_2 \mu_*))
\end{aligned} \tag{51}$$

where $C_3 = C_1(C_D - 1)$. In the third line above, we used that $\tilde{\eta} \circ V^{-1}$ is a non-decreasing function.

By Lemma 3.1 and [8, Theorem 1.5.3], $\tilde{\eta}$ is regularly varying of positive index. Hence by Potter's bounds [8, Theorem 1.5.6] and using that $\tilde{\eta}$ is a positive continuous function, there exists $C_4 > 1$, $\beta_1 > \beta_2 > 0$ such that

$$\frac{\tilde{\eta}(s)}{\tilde{\eta}(t)} \le C_4 \max\left(\left(\frac{s}{t}\right)^{\beta_1}, \left(\frac{s}{t}\right)^{\beta_2} \right) \tag{52}$$

for all $s, t \in [1,\infty)$. By (2), (49) and (52) , we get

$$\tilde{\eta}(V^{-1}(C_D^k C_2 \mu_*)) \le \tilde{\eta}(2^{k-L} V^{-1}(C_D^L C_2 \mu_*)) \le C_4 2^{\beta_2 (k-L)} \tilde{\eta}(V^{-1}(C_D^L C_2 \mu_*)) \tag{53}$$

for all $k = 1, 2, \ldots, L$. By (51) and (53),

$$g(t) \le C_5 \tilde{\eta}(V^{-1}(C_D C_2 / t))$$

for all $t \le \mu_*^{-1}$, where $C_5 := C_3 C_4 (1 - 2^{-\beta_2})^{-1}$. Therefore

$$t = g(m(t)) \le C_5 \tilde{\eta}(V^{-1}(C_D C_2 / m(t)))$$

for all $t \ge 0$.

We use an asymptotic inverse ζ of the function η as described in Lemma 3.1. Hence by Potter's theorem [8, Theorem 1.5.6]) and (40) , there exists $C_6, C_7 > 0$ such that

$$\zeta(t) \le C_6 \zeta(t/C_5) \le C_6 \zeta(\eta(V^{-1}(C_D C_2/m(t)))) \le C_7 V^{-1}(C_D C_2/m(t)) \tag{54}$$

for all $t \ge 1$. By (4), there exists $C_8 > 0$ such that

$$m(t) \le C_D C_2 / V(\zeta(t)/C_7) \le \frac{C_8}{V(\zeta(t))}. \tag{55}$$

The conclusion follows from (50). □

4. Lower bound on ψ_K

The lower bound on ψ_K follows from a test function argument due to Coulhon and Grigor'yan [13, Theorem 4.6]. However we need a good test function for that argument to work. Such a test function can be obtained from the resistance estimate in (8).

Theorem 4.1. *Let (Γ, μ) be a weighted graph satisfying (1), (2), (3), (5) and its heat kernel p_n satisfies the sub-Gaussian bounds (6) and (7) with escape time exponent γ. Let K be a Markov operator symmetric with respect to the measure μ satisfying (J_ϕ), where $\phi : [0,\infty) \to [1,\infty)$ is a continuous regularly varying function of positive index. Then there exists a constant $c > 0$ such that*

$$\psi_K(n) \ge \frac{c}{V(\zeta(n))}$$

for all $n \in \mathbb{N}$, where $\zeta : [0,\infty) \to [1,\infty)$ is a continuous non-decreasing function which is an asymptotic inverse of $t \mapsto t^\gamma / \int_0^t \frac{s^{\gamma-1}\,ds}{\phi(s)}$.

Proof. By Lemma 1.5, we have

$$\frac{\|K^l f\|_2^2}{\|f\|_2^2} \ge \left(\frac{\|Kf\|_2^2}{\|f\|_2^2} \right)^l. \tag{56}$$

For any finite set A define

$$\lambda(A) = \sup_{\substack{\operatorname{supp}(f) \subseteq A, \\ f \not\equiv 0}} \frac{\|Kf\|_2^2}{\|f\|_2^2}.$$

Then by (56) and Cauchy-Schwarz inequality

$$\begin{aligned} \psi_K(n) = \|K^n\|_{1\to 2}^2 &\ge \sup_A \sup_{\substack{\operatorname{supp}(f) \subseteq A, \\ \|f\|_1 = 1}} \|f\|_2^2 \left(\frac{\|Kf\|_2^2}{\|f\|_2^2} \right)^n \\ &\ge \sup_A \frac{\lambda(A)^n}{\mu(A)}. \end{aligned} \tag{57}$$

We write $\lambda(A)$ as

$$\lambda(A) = 1 - (1-\lambda(A)) = 1 - \inf_{\substack{\operatorname{supp}(f)\subseteq A,\\ f\not\equiv 0}} \frac{\mathcal{E}_{K^2}(f,f)}{\|f\|_2^2}. \tag{58}$$

To obtain a lower bound on $\lambda(A)$ it suffices to pick a test function f. By Lemma 1.4, Proposition 2.5, there exists $C_1 > 0$ such that

$$\mathcal{E}_{K^2}(f,f) \le 2\mathcal{E}_K(f,f) \le C_1 \mathcal{E}_{Q_\phi}(f,f) = C_1 c_\phi \sum_{n=1}^\infty \frac{1}{n\phi(n)} \mathcal{E}_{Q^{2\lfloor n^\gamma \rfloor}}(f,f). \tag{59}$$

By Lemma 2.3, (8) and (4), there exist constants $c_1 \in (0,1)$ and $C_2, C_3 > 1$ such that

$$R_Q(B(x,c_1R), B(x,R)^c) \ge C_2^{-1} \frac{R^\gamma}{V(R)}$$

for all $x \in \Gamma$ and for all $R \ge C_3$. Therefore for any $x \in \Gamma$ and for any $R > C_3$, there exists $f \in \mathbb{R}^\Gamma$ satisfying $\operatorname{supp}(f) \subseteq B(x,R)$, $f\big|_{B(x,c_1R)} \equiv 1$ and

$$\mathcal{E}_Q(f,f) \le \frac{2C_2 V(R)}{R^\gamma}. \tag{60}$$

Since such a function has $\|f\|_2^2 \ge V(x,c_1R)$, by (3), (4) and (60), there exists $C_4 > 1$ such that the following holds: for any $x \in \Gamma$ and for any $R > C_3$, there exists $f \in \mathbb{R}^\Gamma$ satisfying $\operatorname{supp}(f) \subseteq B(x,R)$ and

$$\frac{\mathcal{E}_Q(f,f)}{\|f\|_2^2} \le C_4 R^{-\gamma}. \tag{61}$$

Using Lemma 1.4 and the bound $\mathcal{E}_{Q^{2k}(f,f)} = \|f\|_2^2 - \left\|Q^k f\right\|_2^2 \le \|f\|_2^2$, we obtain

$$\sum_{n=1}^\infty \frac{1}{n\phi(n)} \mathcal{E}_{Q^{2\lfloor n^\gamma \rfloor}}(f,f) \le 2 \sum_{n=1}^{\lfloor R \rfloor} \frac{n^{\gamma-1}}{\phi(n)} \mathcal{E}_Q(f,f) + \|f\|_2^2 \sum_{n=\lfloor R \rfloor+1}^\infty \frac{1}{n\phi(n)} \tag{62}$$

for all $f \in \ell^2(\Gamma,\mu)$. For the second term above, we use [8, Proposition 1.5.10] to obtain $C_5 > 0$ such that

$$\sum_{n=\lfloor R \rfloor+1}^\infty \frac{1}{n\phi(n)} \le C_5 \frac{1}{\phi(R)} \tag{63}$$

for all $R \ge 1$. By Potter's bound [8, Theorem 1.5.6] and continuity of ϕ, there exists $C_6 > 0$ such that

$$\sum_{n=1}^{\lfloor R \rfloor} \frac{n^{\gamma-1}}{\phi(n)} \le C_6 \int_0^R \frac{s^{\gamma-1}\,ds}{\phi(s)} \tag{64}$$

for all $R \ge 1$.

Combining (58), (59), (61), (62), (63), (64) and using Lemma 3.1(b), there exist constants $C_7 > 0$ and $R_0 > 0$ such that

$$\lambda(B(x,R)) \ge 1 - \frac{C_7}{\eta(R)}$$

for all $R > R_0$. Combining (57), (3), (40) of Lemma 3.1(c) along with the substitution $R = \zeta(n)$, there exists $N_1, C_8, c_1 > 0$ such that

$$\psi_K(n) \geq \frac{C_h^{-1}}{V(\zeta(n))}\left(1 - \frac{C_7}{\eta(\zeta(n))}\right)^n \geq \frac{C_h^{-1}}{V(\zeta(n))}\left(1 - \frac{C_8}{n}\right)^n \geq \frac{c_1}{V(\zeta(n))}$$

for all $n \in \mathbb{N}$ with $n \geq N_1$. The case $n \leq N_1$ follows from (J_ϕ). □

Proof of Theorem 1.1. The upper bound and lower bound follows from Theorems 3.3 and 4.1 respectively. □

5. Stable subordination and the case $\beta < \gamma$

In this section, we provide evidence to the conjecture in Remark 1(b) and (e). Let (Γ, μ) be a weighted graph satisfying the volume doubling condition: there exists $C_D > 0$ such that

$$V_\mu(x, 2r) \leq C_D V_\mu(x, r) \tag{65}$$

for all $x \in \Gamma$ and for all $r > 0$. By a slight abuse of notation, we denote V_μ by V in this section. Similar to (4), there is a volume comparison estimate

$$\frac{V(x,r)}{V(x,s)} \leq C_D \left(\frac{r}{s}\right)^\alpha \tag{66}$$

for any $x \in \Gamma$, for all $0 < s \leq r$ and for all $\alpha \geq \log_2 C_D$.

As before let P and p_n denote the Markov operator corresponding to the natural random walk and the heat kernel respectively. We assume that the heat kernel satisfies the following sub-Gaussian estimates. There exist constants $c, C > 0$ such that, for all $x, y \in \Gamma$

$$p_n(x,y) \leq \frac{C}{V(x, n^{1/\gamma})} \exp\left[-\left(\frac{d(x,y)^\gamma}{Cn}\right)^{\frac{1}{\gamma-1}}\right], \forall n \geq 1 \tag{67}$$

and

$$(p_n + p_{n+1})(x,y) \geq \frac{c}{V(x, n^{1/\gamma})} \exp\left[-\left(\frac{d(x,y)^\gamma}{cn}\right)^{\frac{1}{\gamma-1}}\right], \forall n \geq 1 \vee d(x,y). \tag{68}$$

Consider a random walk X_n driven by the operator Q defined in (21). We consider the continuous time Markov chain $Y_{\beta_0}(t) = X_{N(S_{\beta_0}(t))}$ where $N(t)$ and S_{β_0} are independent Poisson process and β_0-stable subordinator for some $\beta_0 \in (0,1)$. Let k_{t,β_0} denote the kernel of $Y_{\beta_0}(t)$ with respect to the measure μ. By definition of k_{t,β_0}, we have

$$k_{t,\beta_0}(x,y) = \sum_{i=0}^{\infty} A_{\beta_0}(t,i) q_i(x,y) \tag{69}$$

for all $t \geq 0$ and for all $x, y \in \Gamma$, where $A_{\beta_0}(t,i) := \mathbb{P}(N(S_{\beta_0}(t)) = i)$. Let q_i denote the kernel of the iterated operator Q^i with respect to the measure μ for $i \in \mathbb{N}$. By the same proof as Lemma 2.2, we get similar sub-Gaussian estimates for the more general volume doubling setup. We assume that the kernel q_n satisfies the following sub-Gaussian estimates: There exist constants $c, C > 0$ such that, for all $x, y \in \Gamma$

$$q_n(x,y) \le \frac{C}{V(x,n^{1/\gamma})} \exp\left[-\left(\frac{d(x,y)^\gamma}{Cn} \right)^{\frac{1}{\gamma-1}} \right], \forall n \ge 1 \tag{70}$$

and

$$q_n(x,y) \ge \frac{c}{V(x,n^{1/\gamma})} \exp\left[-\left(\frac{d(x,y)^\gamma}{cn} \right)^{\frac{1}{\gamma-1}} \right], \forall n \ge 1 \vee d(x,y). \tag{71}$$

Using estimates on the stable subordinator S_{β_0} and the estimates on the kernel q_n similar to Lemma 2.2, we show the following:

Theorem 5.1. *Let (Γ, μ) be a weighted graph satisfying (65) and its heat kernel p_n satisfies the sub-Gaussian bounds (67) and (68) with escape time exponent γ. Let k_{t,β_0} be the symmetric Markov kernel with respect to the measure μ defined by (69). Then for all $\beta_0 \in (0,1)$ there exists a constant $C > 0$ such that*

$$k_{n,\beta_0}(x,y) \le C \left(\frac{1}{V(x,n^{1/\beta})} \wedge \frac{n}{V(x,d(x,y))(1+d(x,y))^\beta} \right) \tag{72}$$

and

$$k_{n,\beta_0}(x,y) \ge C^{-1} \left(\frac{1}{V(x,n^{1/\beta})} \wedge \frac{n}{V(x,d(x,y))(1+d(x,y))^\beta} \right) \tag{73}$$

for all $x, y \in \Gamma$ and for all $n \in \mathbb{N}^*$, where $\beta = \beta_0 \gamma$.

We begin by recalling some known estimates for stable subordinator. Let $f_{t,\beta_0}(u)$ be the density of the β_0-stable subordinator $S_{\beta_0}(t)$. We have the scaling relation

$$f_{t,\beta_0}(u) = t^{-1/\beta_0} f_{1,\beta_0}(t^{-1/\beta_0}u), \qquad \beta_0 \in (0,1).$$

By standard estimates on f_{t,β_0} (see [18, Section 3]) there exist constants $c_1, C_1 > 0$ such that

$$f_{t,\beta_0}(u) \le C_1 t u^{-1-\beta_0}, \qquad t, u > 0, \tag{74}$$

$$f_{1,\beta_0}(u) \le C_1 u^{-\frac{2-\beta_0}{2-2\beta_0}} e^{-c_1 u^{-\frac{\beta_0}{1-\beta_0}}}, \qquad u \in (0,1), \tag{75}$$

$$f_{t,\beta_0}(u) \ge c_1 t u^{-1-\beta_0}, \qquad t > 0, u > t^{1/\beta_0}. \tag{76}$$

Next, we estimate the quantity

$$A_{\beta_0}(t,i) = \mathbb{P}(N(S_{\beta_0}(t)) = i) = \int_0^\infty f_{t,\beta_0}(u) \frac{e^{-u}u^i}{i!}\, du. \tag{77}$$

By (74) and Stirling asymptotics for Gamma function, there exists $C_2 > 0$

$$\begin{aligned} A_{\beta_0}(t,i) &\le C_1 \int_0^\infty t u^{-1-\beta_0} \frac{e^{-u}u^i}{i!}\, du \\ &\le C_1 t i^{-1} \frac{\Gamma(i-\beta_0)}{\Gamma(i)} \le C_2 \frac{t}{i^{1+\beta_0}} \end{aligned} \tag{78}$$

for all $t > 0$ and for all $i \in \mathbb{N}^*$. By Chebychev's inequality applied to Gamma distribution, we have

$$\int_{\lambda/2}^{\infty} \frac{e^{-u}u^{\lambda-1}}{\Gamma(\lambda)}\,du \geq \frac{1}{5} \tag{79}$$

for all $\lambda \geq 5$. Therefore, there exists $c_2 > 0$ such that

$$\begin{aligned} A_{\beta_0}(t,i) &\geq c_1 \int_{t^{1/\beta_0}}^{\infty} t u^{-1-\beta_0} \frac{e^{-u}u^i}{i!}\,du \qquad (80)\\ &\geq c_1 \int_{(i-\beta_0)/2}^{\infty} t u^{-1-\beta_0} \frac{e^{-u}u^i}{i!}\,du \\ &\geq \frac{c_1}{5} t \frac{\Gamma(i-\beta_0)}{i\Gamma(i)} \geq c_2 \frac{t}{i^{1+\beta_0}} \qquad (81) \end{aligned}$$

for all $\beta_0 \in (0,1)$, for all $i \in \mathbb{N}^*$ and for all $t > 0$ such that $i \geq \max\left(6, 4t^{1/\beta_0}\right)$. We used (76) in the first line $i \geq \max\left(6, 4t^{1/\beta_0}\right)$ in the second line and (79) and Stirling asymptotics for Gamma function in the last line.

We need the following estimate to prove the desired diagonal upper bound.

Lemma 5.2. *Under the doubing assumption (65), there exists $C_1 > 0$ such that*

$$\sum_{i=0}^{\infty} \frac{\exp(-u)u^i}{i!} \frac{1}{V(x, i^{1/\gamma})} \leq \frac{C_1}{V(x, u^{1/\gamma})} \tag{82}$$

for all $x \in \Gamma$ and for all $u \geq 0$.

Proof. Note that

$$\begin{aligned} \sum_{i=0}^{\infty} \frac{\exp(-u)u^i}{i!} \frac{1}{V(x,i^{1/\gamma})} &\leq \frac{1}{V(x,u^{1/\gamma})} \sum_{i=0}^{\lfloor u \rfloor} \frac{\exp(-u)u^i}{i!} \frac{V(x,u^{1/\gamma})}{V(x,i^{1/\gamma})} + \frac{1}{V(x,u^{1/\gamma})} \\ &\leq \frac{C_2}{V(x,u^{1/\gamma})} \sum_{i=0}^{\infty} \frac{\exp(-u)u^i}{i!} \left(\frac{u}{i+1}\right)^{n_0} + \frac{1}{V(x,u^{1/\gamma})} \\ &\leq \frac{C_2(n_0!)}{V(x,u^{1/\gamma})} \sum_{i=0}^{\infty} \frac{\exp(-u)u^{i+n_0}}{(i+n_0)!} + \frac{1}{V(x,u^{1/\gamma})} \\ &\leq \frac{C_3}{V(x,u^{1/\gamma})} \qquad (83) \end{aligned}$$

where $n_0 = \lceil (\log_2 C_D)/\gamma \rceil$. We used (66) in the second line. $\square$

Proof of Theorem 5.1. We start by showing the off-diagonal lower bound for the case $d(x,y)^\gamma \geq 4n^{1/\beta_0}$. By (69), (81),(66) and (71), we have

$$\begin{aligned} k_{n,\beta_0}(x,y) &\geq c_1 \sum_{i=\lceil d(x,y)^\gamma \rceil}^{2\lceil d(x,y)^\gamma \rceil} \frac{n}{(1+i)^{1+\beta_0}} \frac{1}{V(x,d(x,y))} \\ &\geq c_2 \frac{n}{(1+d(x,y))^\beta} \frac{1}{V(x,d(x,y))} \end{aligned}$$

for all $x, y \in \Gamma$, for all $n \in \mathbb{N}^*$ such that $d(x,y)^\gamma \geq 4n^{1/\beta_0}$. Next, we show the near-diagonal lower bound for the case $d(x,y)^\gamma \leq 4n^{1/\beta_0}$. By (69), (81),(66) and (71), we have

$$k_{n,\beta_0}(x,y) \geq c_3 \sum_{i=\lceil 4n^{1/\beta_0} \rceil}^{\lceil 8n^{1/\beta_0} \rceil} \frac{n}{(1+i)^{1+\beta_0}} \frac{1}{V(x,n^{1/\beta})} \geq \frac{c_4}{V(x,n^{1/\beta})}$$

for $x, y \in \Gamma$ and for all $n \in \mathbb{N}^*$ such that $d(x,y)^\gamma \leq 4n^{1/\beta_0}$.

We prove the diagonal upper bound below. We use (69), (77) and Fubini's theorem to obtain

$$k_{n,\beta_0}(x,y) = \int_0^\infty f_{n,\beta_0}(u) \sum_{i=0}^\infty \frac{e^{-u}u^i}{i!} q_i(x,y)\, du. \tag{84}$$

Combining (70) ,(84) and Lemma 5.2, there exists $C_2, C_3, C_4 > 0$ such that

$$\begin{aligned}
k_{n,\beta_0}(x,y) &\leq C_1 \int_0^\infty f_{n,\beta_0}(u) \sum_{i=0}^\infty \frac{e^{-u}u^i}{i!} \frac{1}{V(x,i^{1/\gamma})}\, du \\
&\leq C_2 \int_0^\infty f_{n,\beta_0}(u) \frac{1}{V(x,u^{1/\gamma})}\, du \\
&= C_2 \int_0^\infty f_{1,\beta_0}(s) \frac{1}{V(x,n^{1/\beta}s^{1/\gamma})}\, ds \\
&\leq \frac{C_2}{V(x,n^{1/\beta})} + \frac{C_3}{V(x,n^{1/\beta})} \int_0^1 s^{-\frac{2-\beta_0}{2-2\beta_0}} e^{-c_1 s^{-\frac{\beta_0}{1-\beta_0}}} \frac{1}{s^{(\log_2 C_D)/\gamma}}\, ds \\
&\leq \frac{C_4}{V(x,n^{1/\beta})}.
\end{aligned} \tag{85}$$

Next, we show the off-diagonal upper bound in (72). Combining (69), (78), (70), there exists $C_5, C_6, C_7 > 0$ such that

$$\begin{aligned}
&k_{n,\beta_0}(x,y) \\
&\leq C_5 n \sum_{i=1}^\infty (1+i)^{-1-\beta_0} \frac{C}{V(x,i^{1/\gamma})} \exp\left[-\left(\frac{d(x,y)^\gamma}{Ci} \right)^{\frac{1}{\gamma-1}} \right] \\
&\leq \frac{C_6 n}{(1+d(x,y))^\beta V(x,d(x,y))} \Bigg((1+d(x,y))^\beta \sum_{i=1+\lfloor d(x,y)^\gamma \rfloor}^\infty (1+i)^{-1-\beta_0} \\
&\quad + d(x,y)^{-\gamma} \sum_{i=1}^{\lfloor d(x,y)^\gamma \rfloor} \left(\frac{d(x,y)^\gamma}{i} \right)^{1+\beta_0+(\alpha/\gamma)} \exp\left[-\left(\frac{d(x,y)^\gamma}{Ci} \right)^{\frac{1}{\gamma-1}} \right] \Bigg) \\
&\leq \frac{C_7 n}{(1+d(x,y))^\beta V(x,d(x,y))}
\end{aligned} \tag{86}$$

for all $x, y \in \Gamma$ with $x \neq y$ and for all $n \in \mathbb{N}$. $\square$

References

[1] M. T. Barlow, *Which values of the volume growth and escape time exponent are possible for a graph?*, Rev. Mat. Iberoamericana 20 (2004), no. 1, 1–31. MR2076770

[2] M. T. Barlow, R. F. Bass, T. Kumagai, *Parabolic Harnack inequality and heat kernel estimates for random walks with long range jumps*, Math. Z. 261 (2009), no. 2, 297–320. MR2457301.

[3] M. T. Barlow, T. Coulhon, T. Kumagai, *Characterization of sub-Gaussian heat kernel estimates on strongly recurrent graphs*, Comm. Pure Appl. Math. 58 (2005), no. 12, 1642–1677. MR2177164

[4] M. T. Barlow, A. Grigor'yan, T. Kumagai, *Heat kernel upper bounds for jump processes and the first exit time*, J. Reine Angew. Math. 626 (2009), 135–157. MR2492992

[5] R. F. Bass, D. A. Levin, *Transition probabilities for symmetric jump processes*, Trans. Amer. Math. Soc. 354 (2002), no. 7, 2933–2953. MR1895210

[6] A. Bendikov, L. Saloff-Coste, *Random walks on groups and discrete subordination*, Math. Nachr. 285 (2012), no. 5-6, 580–605. MR2902834

[7] A. Bendikov, L. Saloff-Coste, *Random walks driven by low moment measures*, Ann. Probab. 40 (2012), no. 6, 2539–2588. MR3050511

[8] N. H. Bingham, C. M. Goldie, and J. L. Teugels, *Regular variation*, Encyclopedia of Mathematics and its Applications, vol. 27, Cambridge University Press, Cambridge, 1987. MR898871

[9] T. Coulhon, *Ultracontractivity and Nash Type Inequalities* J. Funct. Anal. 141 (1996), no. 2, 510–539. MR1418518

[10] Z.-Q. Chen, T. Kumagai, *Heat kernel estimates for stable-like processes on d-sets*, Stochastic Process. Appl. 108 (2003), no. 1, 27–62. MR2008600

[11] Z.-Q. Chen, T. Kumagai, *Heat kernel estimates for jump processes of mixed types on metric measure spaces*, Probab. Theory Related Fields 140 (2008), no. 1-2, 277–317. MR2357678

[12] E. A. Carlen, S. Kusuoka, D. W. Stroock, *Upper bounds for symmetric Markov transition functions*, Ann. Inst. H. Poincaré Probab. Statist. 23 (1987), no. 2, suppl., 245–287. MR0898496

[13] T. Coulhon, A. Grigor'yan, *On-diagonal lower bounds for heat kernels and Markov chains*, Duke Math. J. 89 (1997), no. 1, 133–199. MR1458975

[14] T. Delmotte, *Parabolic Harnack inequality and estimates of Markov chains on graphs*, Rev. Mat. Iberoamericana 15 (1999), no. 1, 181–232. MR1681641

[15] M. Murugan, L. Saloff-Coste, *Transition probability estimates for long range random walks*, arXiv:1411.2706

[16] Ch. Pittet, L. Saloff-Coste, *On the stability of the behavior of random walks on groups*, J. Geom. Anal. 10 (2000), no. 4, 713–737. MR1817783

[17] Ch. Pittet, L. Saloff-Coste, *A survey on the relationships between volume growth, isoperimetry and the behavior of simple random walk on Cayley graphs, with examples* (in preparation)

[18] P. Graczyk, A. Stós, *Transition density estimates for stable processes on symmetric spaces*, Pacific J. Math. 217 (2004), no. 1, 87–100. MR2105767

[19] A. Grigor'yan, J. Hu, K.-S. Lau, *Heat kernels on metric spaces with doubling measure*, Fractal geometry and stochastics IV, 3–44, Progr. Probab., 61, Birkhäuser Verlag, Basel, 2009. MR2762672

[20] A. Grigor'yan, A. Telcs, *Sub-Gaussian estimates of heat kernels on infinite graphs*, Duke Math. J. 109 (2001), no. 3, 451–510. MR1853353

[21] A. Grigor'yan, A. Telcs, *Harnack inequalities and sub-Gaussian estimates for random walks*, Math. Ann. 324 , 551-556 (2002). MR1938457

[22] T. Kumagai, *Anomalous random walks and diffusions: From fractals to random media*, Proceedings of 2014 ICM, Seoul.

[23] L. Saloff-Coste, *A note on Poincaré, Sobolev, and Harnack inequalities*, Internat. Math. Res. Notices 1992, no. 2, 27–38. MR1150597

[24] L. Saloff-Coste, T. Zheng, *On some random walks driven by spread-out measures*, arXiv:1309.6296 .

Acknowledgements. We thank the referee for helpful comments and careful reading of the manuscript.

Department of Mathematics, University of British Columbia and Pacific Institute for the Mathematical Sciences, Vancouver, BC V6T 1Z2, Canada.
E-mail address, M. Murugan: mathav@math.ubc.ca

Department of Mathematics, Cornell University, Ithaca, NY 14853, USA
E-mail address, L. Saloff-Coste: lsc@math.cornell.edu