Invent. math.
https://doi.org/10.1007/s00222-022-01148-3

On the conformal walk dimension: quasisymmetric uniformization for symmetric diffusions

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Received: 1 May 2021 / Accepted: 26 July 2022

Abstract We introduce the notion of conformal walk dimension, which serves as a bridge between elliptic and parabolic Harnack inequalities. The importance of this notion is due to the fact that, for a given strongly local, regular symmetric Dirichlet space in which every metric ball has compact closure (*MMD space*), the finiteness of the conformal walk dimension characterizes the conjunction of the metric doubling property and the elliptic Harnack inequality. Roughly speaking, the conformal walk dimension of an MMD space is defined as the infimum over all possible values of the walk dimension with which the parabolic Harnack inequality can be made to hold by a time change

Naotaka Kajino was partially supported by JSPS KAKENHI Grant Numbers JP17H02849, JP18H01123.
Mathav Murugan was partially supported by NSERC and the Canada research chairs program.

This work was supported by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University.

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Published online: 08 November 2022

of the associated diffusion and by a quasisymmetric change of the metric. We show that the conformal walk dimension of any MMD space satisfying the metric doubling property and the elliptic Harnack inequality is two, and provide a necessary condition for a pair of such changes to attain the infimum defining the conformal walk dimension when it is attained by the original pair. We also prove a necessary condition for the existence of such a pair attaining the infimum in the setting of a self-similar Dirichlet form on a self-similar set, and apply it to show that the infimum fails to be attained for the Vicsek set and the N-dimensional Sierpiński gasket with $N \geq 3$, in contrast to the attainment for the two-dimensional Sierpiński gasket due to Kigami (Math Ann 340(4):781–804, 2008).

Mathematics Subject Classification Primary 30L10, 31C25, 31E05, 35K08; Secondary 28A80, 60G30, 60J46, 60J60

Contents

1 Introduction

What is the "best" way to parametrize a space? This vaguely stated question is the motivation for our work and several earlier works. By a parametrization, we mean a bijection $f : X \to M$ between the given space X and another "model space" M with more desirable properties. For example, the Riemann mapping theorem (or more generally, the uniformization theorem for Riemann surfaces) and geometric flows like the Ricci flow can be viewed as an attempt to answer the above question. In the Riemann mapping theorem example, X is a proper simply connected domain in $\mathbb{C}$, M is the unit disk, and f is a conformal map. In the Ricci flow example, X is a Riemannian manifold, M is a Riemannian manifold with constant Ricci curvature, and f is a diffeomorphism. This work aims to formulate and answer this question for spaces satisfying Harnack inequalities. In this work, X is a space equipped with a symmetric diffusion that satisfies the elliptic Harnack inequality, M satisfies the stronger parabolic Harnack inequality and f is a quasisymmetry along with a time change of the diffusion on X (quasisymmetry is an analogue of conformal maps for metric spaces).

This paper uses quasiconformal geometry and time change of diffusion processes to understand the relationship between elliptic and parabolic Harnack inequalities. The analysis using quasiconformal geometry also leads to a natural uniformization problem for spaces satisfying the elliptic Harnack inequality. Our results can be viewed as a bridge between analysis in smooth and fractal spaces and also as a bridge between elliptic and parabolic Harnack inequalities.

We informally describe the setup and results. A more precise treatment is given in Sect. 2. The setup of this work is a metric space equipped with a Radon measure m with full support and an m-symmetric diffusion process. Equivalently, we consider a metric space (X, d) equipped with such m and a strongly local, regular symmetric Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(X, m)$. We always assume that $B(x, r) := B_d(x, r) := \{y \in X \mid d(x, y) < r\}$ has compact closure in X for any $(x, r) \in X \times (0, \infty)$ and that X contains at least two elements, and call $(X, d, m, \mathcal{E}, \mathcal{F})$ a *metric measure Dirichlet space* or an *MMD space* for short. Associated to an MMD space $(X, d, m, \mathcal{E}, \mathcal{F})$ is a non-negative self-adjoint operator $\mathcal{L}$ on $L^2(X, m)$ such that the corresponding Markov semigroup $(P_t)_{t \geq 0}$ is given by $P_t = e^{-t\mathcal{L}}$. The operator $\mathcal{L}$ is called the *generator* of $(X, d, m, \mathcal{E}, \mathcal{F})$, which is an analog of the Laplace operator in the abstract setting of MMD spaces. We refer to [24,33] for the theory of Dirichlet forms.

We recall that this setup includes Brownian motion on a Riemannian manifold, where d is the Riemannian distance function, m is the Riemannian measure, $\mathcal{F}$ is the Sobolev space $W^{1,2}$, and $\mathcal{E}(f, f) = \int |\nabla f|^2 \, dm$, where

∇ denotes the Riemannian gradient. In this case, the corresponding generator $\mathcal{L}$ is the Laplace–Beltrami operator with a minus sign (so that $\mathcal{L}$ is non-negative definite). This setup also covers non-smooth settings like diffusions on fractals including the Sierpiński gasket and the Sierpiński carpet. We refer the reader to [3] for an introduction to diffusions on fractals. Random walks on graphs can also be studied in this framework because the corresponding cable processes share many properties with the original random walks (see [6] for this approach).

An MMD space has an associated sheaf of *harmonic* and *caloric* functions. Roughly speaking, harmonic and caloric functions are generalization of solutions to the "Laplace equation" $\Delta h \equiv 0$ and the "heat equation" $\partial_t u - \Delta u \equiv 0$, respectively. Let $\mathcal{L}$ denote the generator of an MMD space $(X, d, m, \mathcal{E}, \mathcal{F})$. Let $h : U \to \mathbb{R}$ be a measurable function in an open set U. We say that h is harmonic in U, if it satisfies $\mathcal{L}h \equiv 0$ in U interpreted in a weak sense. Similarly, we say that a space-time function $u : (a, b) \times U \to \mathbb{R}$ is caloric in $(a, b) \times U$ if it satisfies the "heat equation" $\partial_t u + \mathcal{L}u \equiv 0$ interpreted in a weak sense.

Harnack inequalities are fundamental regularity estimates that have numerous applications in partial differential equations and probability theory. We refer to [57] for a nice survey on Harnack inequality and its variants. We recall the (scale-invariant) elliptic and parabolic Harnack inequalities. We say that an MMD space $(X, d, m, \mathcal{E}, \mathcal{F})$ satisfies the *elliptic Harnack inequality* (abbreviated as EHI), if there exist $C > 1$ and $\delta \in (0, 1)$ such that for all $x \in X$, $r > 0$ and for any non-negative harmonic function h on the ball $B(x, r)$, we have

$$\operatorname*{ess\,sup}_{B(x,\delta r)} h \leq C \operatorname*{ess\,inf}_{B(x,\delta r)} h. \tag{EHI}$$

We say that an MMD space $(X, d, m, \mathcal{E}, \mathcal{F})$ satisfies the *parabolic Harnack inequality* with walk dimension $\beta > 0$ (abbreviated as PHI(β)), if there exist $0 < C_1 < C_2 < C_3 < C_4 < \infty$, $C_5 > 1$ and $\delta \in (0, 1)$ such that for all $x \in X$, $r > 0$ and for any non-negative bounded caloric function u on the space-time cylinder $Q = (a, a + C_4 r^\beta) \times B(x, r)$, we have

$$\operatorname*{ess\,sup}_{Q_-} u \leq C_5 \operatorname*{ess\,inf}_{Q_+} u, \tag{PHI(β)}$$

where $Q_- = (a + C_1 r^\beta, a + C_2 r^\beta) \times B(x, \delta r)$ and $Q_+ = (a + C_3 r^\beta, a + C_4 r^\beta) \times B(x, \delta r)$.

We briefly review some earlier works on Harnack inequalities, referring the reader to [57] for a more detailed survey of the literature. In a series of celebrated works, Moser showed EHI and PHI(2) for uniformly elliptic divergence form operators on $\mathbb{R}^n$ [77,78]. Cheng, Li and Yau obtained gradient estimates that imply EHI and PHI(2) for Riemannian manifolds with non-negative Ricci curvature [25,72,92]. Grigor'yan and Saloff-Coste independently characterized PHI(2) using the volume doubling property and the Poincaré inequality [35,82]. This characterization was extended to PHI(β) by Barlow, Bass and Kumagai [6,7]. A similar characterization of the simpler elliptic Harnack inequality remained open until recently [9,11,12].

Note that every harmonic function lifts to a caloric function. More precisely, if h is harmonic on $B(x,r)$, then $u(t,x) = h(x)$ is caloric on $(a,b) \times B(x,r)$ for all $b > a$. This lift immediately shows that[1]

$$\text{PHI}(\beta) \implies \text{EHI}, \quad \text{for all } \beta > 0. \tag{1.1}$$

However, the converse of the above implication fails. Indeed, Delmotte has constructed an example of a space that satisfies EHI but fails to satisfy PHI(β) for any $\beta > 0$ [31] (see also [9]). Nevertheless, one can characterize the elliptic Harnack inequality in terms of the parabolic Harnack inequality [9,11].

The main idea behind the characterization of EHI is to reparametrize the space and time of the associated diffusion process so that it satisfies PHI(β) for some $\beta > 0$. In the theory of regular symmetric Dirichlet forms the *Revuz correspondence* provides a bijection between the time changes of the process and the family of *smooth measures*. Roughly speaking, smooth measures are Radon measures that do not charge any set of capacity zero. If μ is a smooth measure for an MMD space $(X, d, m, \mathcal{E}, \mathcal{F})$, then it defines a "time-changed" Dirichlet space $(X, d, \mu, \mathcal{E}^{\mu}, \mathcal{F}^{\mu})$ as well as a "time-changed" Markov process. We say that a measure μ is *admissible*, if μ is a smooth measure and has full quasi-support for the Dirichlet form $(\mathcal{E}, \mathcal{F})$, which amounts to saying that μ represents a "time change" keeping the form $\mathcal{E}$ essentially unchanged (see Definitions 2.8 and 2.9). We denote the collection of admissible measures by $\mathcal{A}(X, d, m, \mathcal{E}, \mathcal{F})$. Next, we recall the definition of conformal gauge.

Definition 1.1 (Conformal gauge) Let (X, d) be a metric space and θ be another metric on X. We say that d is *quasisymmetric* to θ, if there exists a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ such that

$$\frac{\theta(x,a)}{\theta(x,b)} \le \eta\left(\frac{d(x,a)}{d(x,b)}\right) \quad \text{for all triples of points } x, a, b \in X, x \neq b.$$

[1] To be precise, while for various purposes it is convenient to formulate EHI without assuming the boundedness of non-negative harmonic functions h, we follow for simplicity the formulation of PHI(β) in [10, Subsection 3.1] which a priori requires the boundedness of non-negative caloric functions u, and then (1.1) is obvious only under the extra assumption of the boundedness of h. It turns out that this extra assumption can be dropped, but the proof of this fact is non-trivial; see Theorem 4.5 and its proof for details.

The *conformal gauge* of a metric space (X, d) is defined as

$$\mathcal{J}(X,d) := \{\theta\colon X \times X \to [0,\infty) \mid \theta \text{ is a metric on } X, \\ d \text{ is quasisymmetric to } \theta\}. \quad (1.2)$$

By [40, Proposition 10.6], being quasisymmetric is an equivalence relation among metrics. That is,

$$d \in \mathcal{J}(X,d) \qquad \text{and} \qquad \mathcal{J}(X,\theta) = \mathcal{J}(X,d) \quad \text{for all } \theta \in \mathcal{J}(X,d). \quad (1.3)$$

The notion of quasisymmetry is an extension of conformal map to the context of metric spaces. Quasisymmetric maps on the real line were introduced by Beurling and Ahlfors, and were studied as boundary values of quasiconformal self-maps of the upper half-plane [14]. The above definition on general metric spaces is due to Tukia and Väisälä [90]. This is the reason behind the terminology “conformal gauge”. We refer to [40,42] for expositions of the theory of quasisymmetric maps and quasiconformal geometry on metric spaces.

To characterize the elliptic Harnack inequality, we reparametrize the space by choosing a new metric in the conformal gauge of (X, d) and we reparametrize time by choosing a new symmetric measure that is admissible. More precisely, given an MMD space $(X, d, m, \mathcal{E}, \mathcal{F})$ satisfying EHI, we seek to find a metric $\theta \in \mathcal{J}(X, d)$ and a measure $\mu \in \mathcal{A}(X, d, m, \mathcal{E}, \mathcal{F})$ such that the corresponding time-changed MMD space $(X, \theta, \mu, \mathcal{E}^\mu, \mathcal{F}^\mu)$ equipped with the new metric θ satisfies PHI(β) for some $\beta > 0$. In other words, we seek to *upgrade* EHI to PHI(β) by reparametrizing space and time. This motivates the notion of conformal walk dimension.

Definition 1.2 (Conformal walk dimension) The *conformal walk dimension* d_{cw} of an MMD space $(X, d, m, \mathcal{E}, \mathcal{F})$ is defined as

$$d_{\mathrm{cw}} := \inf\left\{\beta > 0 \;\middle|\; \begin{array}{l} \text{there exist } \mu \in \mathcal{A}(X,d,m,\mathcal{E},\mathcal{F}) \text{ and } \theta \in \mathcal{J}(X,d) \\ \text{such that } (X,\theta,\mu,\mathcal{E}^\mu,\mathcal{F}^\mu) \text{ satisfies PHI}(\beta) \end{array}\right\}, \quad (1.4)$$

where $\inf \emptyset := \infty$ and $(\mathcal{E}^\mu, \mathcal{F}^\mu)$ denotes the time-changed Dirichlet form on $L^2(X, \mu)$.

We remark that, if $(X, d, m, \mathcal{E}, \mathcal{F})$ satisfies PHI(β), then it is easy to see that for any $\alpha \in (0, 1]$ the MMD space $(X, d^\alpha, m, \mathcal{E}, \mathcal{F})$ satisfies PHI(β/α) and $d^\alpha \in \mathcal{J}(X, d)$. This shows that it is easy to increase the walk dimension by changing the metric to a different one in the conformal gauge, but it is non-trivial to decrease the walk dimension. This explains the “infimum” in (1.4). Another remark, which is based on an observation in [44, Section 1], is that the lower bound

$$d_{\mathrm{cw}} \geq 2 \tag{1.5}$$

is essentially known to experts; indeed, (1.5) can be obtained from the so-called Varadhan-type Gaussian off-diagonal asymptotics of the associated Markov semigroup due to [2, Theorem 2.7] combined with the characterization of PHI(β) for $\beta > 1$ by the *volume doubling property* VD and the *heat kernel estimates* HKE(β) with walk dimension β (see Definitions 3.17, 4.1 and Theorem 4.5).[2]

Two natural questions arise. What is the value of d_{cw}? When is the infimum in (1.4) attained? The answer to the first question is given below. We assume that our metric space (X, d) satisfies the *metric doubling property*, i.e., admits $N \in \mathbb{N}$ such that for all $x \in X$ and $r > 0$, the ball $B(x, r)$ can be covered by N balls of radii $r/2$.

Our first main result (Theorem 2.10) is that the value of the conformal walk dimension is always two, i.e., we always have the equality in (1.5), for any MMD space satisfying the metric doubling property and EHI . In other words, we have the equivalence among the following three conditions, sharpening the existing characterization of EHI (more precisely, its conjunction with the metric doubling property):

(a) $(X, d, m, \mathcal{E}, \mathcal{F})$ satisfies the metric doubling property and EHI .
(b) $d_{\mathrm{cw}} < \infty$.
(c) $d_{\mathrm{cw}} = 2$.

The equivalence between (a) and (b) is contained in [9,11]. That (c) implies (b) is obvious. Our contribution to the above equivalence is the proof that (a) implies (c). Therefore our result sharpens the characterization of EHI in [9,11]. The result that (a) implies (c) is particularly interesting on fractals as we explain below. Diffusions on many regular fractals are known to satisfy PHI(β) with $\beta > 2$. These are often called *anomalous diffusions* to distinguish them from the classical smooth settings like the Euclidean space where one often has Gaussian space-time scaling and PHI(2). However, by the above equivalence one can "improve" from PHI(β) to PHI($2 + \varepsilon$) for any

[2] Since $(\mathcal{E}, \mathcal{F})$ satisfies the locality assumption in [2] by its strong locality and [24, Theorem 2.4.3], we can apply [2, Theorem 2.7], which the conjunction of VD and HKE(β) with $\beta \in (1, 2)$ would contradict in view of the finiteness of the limit in [2, Theorem 2.7] implied by [2, Proposition 5.1] and HKE(β). Therefore the conjunction of VD and HKE(β) for $\beta \in (1, 2)$ cannot hold, hence neither can PHI(β/α) for any $\beta \in (0, 2)$ and any $\alpha \in (\beta/2, \beta) \cap (0, 1]$ by Theorem 4.5 and thus neither can PHI(β) for any $\beta \in (0, 2)$ by the previous remark, proving (1.5).
In Lemma 4.7, we will give an alternative proof of (1.5) based on a relatively simple result from [80].

$\varepsilon > 0$ even on fractals. So this result serves as a bridge between anomalous space-time scaling in fractals and Gaussian space-time scaling seen in smooth settings.

It is worth mentioning that the proof that (a) implies (b) in [9,11] does not give a universal upper bound for d_{cw}. The bound on d_{cw} obtained there depends on the constants in EHI and could be arbitrarily large. To improve the previous (a)-implies-(b) result to the (a)-implies-(c) result, we need a new construction of metrics and measures.

We briefly discuss this new construction in the proof that (a) implies (c), which we will achieve in Sect. 4. The inspiration behind our argument is the uniformization theorem for Riemann surfaces. In the proof of the uniformization theorem, the Green's function of a Riemann surface (or a subset of the surface) plays an essential role in constructing the uniformizing map [76, Chapter 15]. We use certain cutoff functions across annuli with small Dirichlet energy at different scales and locations as a substitute for the Green's function. It is helpful to think of these cutoff functions as equilibrium potentials across annuli. Roughly speaking, the diameter of a ball under the new metric $\theta \in \mathcal{J}(X, d)$ for our construction is proportional to the average gradient of the equilibrium potential chosen at a suitable location and scale.

On a technical level, our proof relies heavily on the theory of Gromov hyperbolic spaces. We view X as the boundary of a Gromov hyperbolic space called the hyperbolic filling. The conformal gauge of X is essentially in a bijective correspondence to the bi-Lipschitz changes of the metric on the hyperbolic filling. A desired bi-Lipshitz change of the metric on the hyperbolic filling is constructed using equilibrium potentials as described above. A major ingredient in the proof is a combinatorial description of the conformal gauge due to Carrasco Piaggio [23], which we will adapt for our purpose in Sect. 3.

Our first main result described above (Theorem 2.10) is a partial converse to the trivial implication $\mathrm{PHI}(\beta) \Longrightarrow \mathrm{EHI}$ in (1.1). The equivalence between (a) and (c) clarifies the extent to which the converse of this trivial implication holds. Although the value of d_{cw} has a simple description, the following questions remain open in general.

Problem 1.3 Given an MMD space $(X, d, m, \mathcal{E}, \mathcal{F})$ that satisfies the metric doubling property and EHI :

(1) (Attainment problem) Determine whether the infimum in (1.4) is attained.
(2) (Gaussian uniformization problem) Describe all the pairs (θ, μ) of metrics $\theta \in \mathcal{J}(X, d)$ and measures $\mu \in \mathcal{A}(X, d, m, \mathcal{E}, \mathcal{F})$ such that the corresponding time-changed MMD space $(X, \theta, \mu, \mathcal{E}^{\mu}, \mathcal{F}^{\mu})$ satisfies PHI(2).

We describe two examples of self-similar fractals for which a positive answer to the attainment problem is known. Kigami has shown in [62] that the MMD space corresponding to the Brownian motion on the two-dimensional

Sierpiński gasket attains the infimum, where μ is the Kusuoka measure and θ is the associated intrinsic metric. Further examples of admissible measures that attain the infimum for the two-dimensional Sierpiński gasket is described in [52]; see Theorem 6.33 below and the references in its proof for details. In retrospect, Kigami's measurable Riemannian structure on the Sierpiński gasket is the first evidence towards the implication (a) $\Longrightarrow$ (c) in Theorem 2.10. Another example of a fractal that attains the infimum in (1.4) is the two-dimensional snowball described in [79]. The "snowball" fractal can be viewed as a limit of Riemann surfaces and is a two-dimensional analog of the von Koch snowflake. In this example, the answer to the attainment problem is obtained by considering a limit of uniformizing maps to $\mathbb{S}^2$ and using the conformal invariance of Brownian motion. Our terminology "Gaussian uniformization problem" is inspired by this example and the classical fact from [89] (see also Proposition 2.11 and Theorem 4.5 below) that PHI(2) is equivalent to Gaussian heat kernel estimates.

Nevertheless, the infimum in (1.4) need not be attained in general. We show in Sect. 6.3 that the Vicsek set and the N-dimensional Sierpiński gasket with $N \geq 3$ fail to attain the infimum in (1.4). The examples with non-attainment of d_{cw} rely on the following result (Theorems 6.16 and 6.54). For a "regular" self-similar fractal, if the infimum in (1.4) is attained by a quasisymmetric metric θ and an admissible measure μ, then it is possible to choose μ as the energy measure of a function that is harmonic outside a canonical boundary. This result immediately implies the non-attainment of d_{cw} for the Vicsek set, since the energy measure of any such harmonic function fails to have full support. The non-attainment of d_{cw} for the N-dimensional Sierpiński gasket with $N \geq 3$ requires a more delicate analysis of the intrinsic metric (see Definition 2.3) associated to the energy measure.

Next, we mention some progress towards the Gaussian uniformization problem. If $(X, \theta, \mu, \mathcal{E}^{\mu}, \mathcal{F}^{\mu})$ satisfies PHI(2), then it easily follows by combining the results in [56] and [80] (see Proposition 2.11-(a)) that the metric θ is bi-Lipschitz equivalent to the intrinsic metric of $(X, \theta, \mu, \mathcal{E}^{\mu}, \mathcal{F}^{\mu})$ and in particular that the metric θ is determined by the measure μ up to a bi-Lipschitz change. Therefore, in order to find a metric $\theta \in \mathcal{J}(X, d)$ and a measure $\mu \in \mathcal{A}(X, d, m, \mathcal{E}, \mathcal{F})$ in the Gaussian uniformization problem, it is enough to find an appropriate measure μ. Furthermore, by [56, Propositions 4.5 and 4.7], we know that any such μ is a minimal energy-dominant measure, i.e., mutually absolutely continuous with respect to the whole family of energy measures (see Definition 2.2), so that any two admissible measures that arise in the Gaussian uniformization problem are mutually absolutely continuous. We strengthen this result by showing in Sect. 5 that any two admissible measures that arise in the Gaussian uniformization problem are A_{∞}-related in (X, d) in the sense of Muckenhoupt (Theorem 2.12). For the MMD space corresponding to the Brownian motion on $\mathbb{R}^n$, we also prove that the measures A_∞-related to the Lebesgue measure are the only ones arising in the Gaussian uniformization problem for $n = 1$ (Theorem 5.18) but are not for $n \geq 2$ (Example 5.14).

This last result on A_∞-relation between admissible measures in the Gaussian uniformization problem and its proof are inspired by a similar result for Ahlfors regular conformal dimension on Loewner spaces [42, Theorem 7.11]. The combinatorial description of conformal gauge used in the proof of Theorem 2.10 was developed in [23] for studying Ahlfors regular conformal dimension. Therefore, we find it appropriate to recall the definition of Ahlfors regular conformal dimension and discuss some related questions.

Given a metric space (X, d) and a Borel measure μ on X, we say that μ is *p-Ahlfors regular* if there exists $C > 0$ such that

$$C^{-1}r^p \leq \mu(B(x,r)) \leq Cr^p \qquad \text{for all } x \in X \text{ and } r > 0 \\ \text{such that } B(x,r) \neq X.$$

It is easy to verify that if a p-Ahlfors regular Borel measure μ exists on (X, d), then the p-dimensional Hausdorff measure $\mathcal{H}^p$ is also p-Ahlfors regular and the Hausdorff dimension of (X, d) is p. Therefore, the existence of a p-Ahlfors regular measure is a property of the metric d. The *Ahlfors regular conformal dimension* of a metric space (X, d) is defined as

$$d_{\mathrm{ARC}}(X,d) = \inf\left\{ p > 0 \;\middle|\; \begin{array}{l} \text{there exist } \theta \in \mathcal{J}(X,d) \text{ and a } p\text{-Ahlfors} \\ \text{regular Borel measure } \mu \text{ on } (X,\theta) \end{array} \right\}. \tag{1.6}$$

The attainment problem for the Ahlfors regular conformal dimension of the standard (two-dimensional) Sierpiński carpet is a well-known open question [17, Problem 6.2]. An important motivation for studying this attainment problem is Cannon's conjecture in geometric group theory. Cannon's conjecture states that every finitely generated, Gromov-hyperbolic group G whose boundary (in the sense of Gromov) is homeomorphic to the 2-sphere is isomorphic to a Kleinian group, i.e., a discrete group of Möbius transformations on the Riemann sphere. Bonk and Kleiner have shown that Cannon's conjecture is equivalent to the attainment of Ahlfors regular conformal dimension of the boundary of such a group [17, Theorem 1.1]. Our results and proof techniques will make it clear that there are similarities between the attainment problems for Ahlfors regular conformal dimension and conformal walk dimension. We hope that some of the methods we develop towards the attainment problem for conformal walk dimension will have applications to the analogous attainment problem for Ahlfors regular conformal dimension.

Our work suggests that it would be useful to develop a theory of non-linear Dirichlet forms to study Ahlfors regular conformal dimension of fractals. In particular, Theorems 6.16 and 6.54 show that if the infimum in (1.4) is attained on a self-similar fractal, then an optimal admissible measure can be chosen to be the energy measure of a harmonic function. This result and its proof suggest that one might be able to construct an optimal Ahlfors regular measure attaining the Ahlfors regular conformal dimension as the "energy measure" of a p-harmonic function. However, the notions of energy measure and p-harmonic functions for non-linear Dirichlet energy remain to be developed on fractals (non-linear Dirichlet energy can be formally viewed as $\int |\nabla f|^p$ with $p \neq 2$ and the corresponding p-harmonic functions can be viewed as minimizers of the non-linear Dirichlet energy). There is a well-developed non-linear potential theory in smooth settings (see [41] and references therein), but a similar theory is yet to be developed on fractals.

Notation 1.4 Throughout this paper, we use the following notation and conventions.

(a) The symbols $\subset$ and $\supset$ for set inclusion *allow* the case of the equality.
(b) For $[0, \infty]$-valued quantities A and B, we write $A \lesssim B$ to mean that there exists an implicit constant $C \in [1, \infty)$ depending on some unimportant parameters such that $A \leq CB$. We write $A \asymp B$ if $A \lesssim B$ and $B \lesssim A$.
(c) $\mathbb{N} := \{n \in \mathbb{Z} \mid n > 0\}$, i.e., $0 \notin \mathbb{N}$.
(d) The cardinality (the number of elements) of a set A is denoted by $\#A \in \mathbb{N} \cup \{0, \infty\}$.
(e) We set $a \vee b := \max\{a, b\}$, $a \wedge b := \min\{a, b\}$, $a^+ := a \vee 0$ and $a^- := -(a \wedge 0)$ for $a, b \in [-\infty, \infty]$, and we use the same notation also for $[-\infty, \infty]$-valued functions and equivalence classes of them. All numerical functions in this paper are assumed to be $[-\infty, \infty]$-valued.
(f) Let X be a non-empty set. We define $\mathbb{1}_A = \mathbb{1}_A^X \in \mathbb{R}^X$ for $A \subset X$ by $\mathbb{1}_A(x) := \mathbb{1}_A^X(x) := \begin{cases} 1 \text{ if } x \in A, \\ 0 \text{ if } x \notin A. \end{cases}$
(g) For a topological space X, we set $C(X) := \{f : X \to \mathbb{R} \mid f \text{ is continuous}\}$ and $C_c(X) := \{f \in C(X) \mid X \setminus f^{-1}(0) \text{ has compact closure in } X\}$.
(h) Let $(X, \mathcal{B})$ be a measurable space and let μ, ν be σ-finite measures on $(X, \mathcal{B})$. We write $\nu \ll \mu$ to mean that ν is absolutely continuous with respect to μ, and $\nu \leq \mu$ to mean that $\nu(A) \leq \mu(A)$ for any $A \in \mathcal{B}$, or equivalently, $\nu \ll \mu$ and $d\nu/d\mu \leq 1$ μ-a.e.

2 Framework and results

In this section, we recall the background definitions and state our main results for general MMD spaces. Our other main results on the attainment problem for self-similar sets are treated separately in Sect. 6.

2.1 Metric measure Dirichlet space and energy measure

Throughout this paper, we consider a metric space (X, d) in which $B(x,r) := B_d(x,r) := \{y \in X \mid d(x,y) < r\}$ is relatively compact (i.e., has compact closure) in X for any $(x,r) \in X \times (0,\infty)$, and a Radon measure m on X with full support, i.e., a Borel measure m on X which is finite on any compact subset of X and strictly positive on any non-empty open subset of X. We always assume that X contains at least two elements, and such a triple (X, d, m) is referred to as a *metric measure space*. We set $\overline{B}(x,r) := \overline{B}_d(x,r) := \{y \in X \mid d(x,y) \le r\}$ for $(x,r) \in X \times (0,\infty)$ and $\mathrm{diam}_d(A) := \mathrm{diam}(A,d) := \sup_{x,y\in A} d(x,y)$ for $A \subset X$ ($\sup\emptyset := 0$).

Furthermore let $(\mathcal{E}, \mathcal{F})$ be a *symmetric Dirichlet form* on $L^2(X,m)$; by definition, $\mathcal{F}$ is a dense linear subspace of $L^2(X,m)$, and $\mathcal{E} : \mathcal{F} \times \mathcal{F} \to \mathbb{R}$ is a non-negative definite symmetric bilinear form which is *closed* ($\mathcal{F}$ is a Hilbert space under the inner product $\mathcal{E}_1 := \mathcal{E} + \langle \cdot, \cdot \rangle_{L^2(X,m)}$) and *Markovian* ($f^+ \wedge 1 \in \mathcal{F}$ and $\mathcal{E}(f^+ \wedge 1, f^+ \wedge 1) \le \mathcal{E}(f,f)$ for any $f \in \mathcal{F}$). Recall that $(\mathcal{E}, \mathcal{F})$ is called *regular* if $\mathcal{F} \cap C_\mathrm{c}(X)$ is dense both in $(\mathcal{F}, \mathcal{E}_1)$ and in $(C_\mathrm{c}(X), \|\cdot\|_{\sup})$, and that $(\mathcal{E}, \mathcal{F})$ is called *strongly local* if $\mathcal{E}(f,g) = 0$ for any $f, g \in \mathcal{F}$ with $\mathrm{supp}_m[f]$, $\mathrm{supp}_m[g]$ compact and $\mathrm{supp}_m[f - a\mathbb{1}_X] \cap \mathrm{supp}_m[g] = \emptyset$ for some $a \in \mathbb{R}$. Here, for a Borel measurable function $f : X \to [-\infty, \infty]$ or an m-equivalence class f of such functions, $\mathrm{supp}_m[f]$ denotes the support of the measure $|f|\,dm$, i.e., the smallest closed subset F of X with $\int_{X\setminus F} |f|\,dm = 0$, which exists since X has a countable open base for its topology; note that $\mathrm{supp}_m[f]$ coincides with the closure of $X \setminus f^{-1}(0)$ in X if f is continuous. The pair $(X, d, m, \mathcal{E}, \mathcal{F})$ of a metric measure space (X, d, m) and a strongly local, regular symmetric Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(X,m)$ is termed a *metric measure Dirichlet space*, or an *MMD space* in abbreviation. We refer to [24,33] for details of the theory of symmetric Dirichlet forms.

We next recall the definition of energy measure and some relevant notions. Note that $fg \in \mathcal{F}$ for any $f, g \in \mathcal{F} \cap L^\infty(X,m)$ by [33, Theorem 1.4.2-(ii)] and that $\{(-n) \vee (f \wedge n)\}_{n=1}^\infty \subset \mathcal{F}$ and $\lim_{n\to\infty} (-n) \vee (f \wedge n) = f$ in norm in $(\mathcal{F}, \mathcal{E}_1)$ by [33, Theorem 1.4.2-(iii)].

Definition 2.1 ([33, (3.2.13), (3.2.14) and (3.2.15)]) Let $(X, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space. The *energy measure* $\Gamma(f,f)$ of $f \in \mathcal{F}$ associated with $(X, d, m, \mathcal{E}, \mathcal{F})$ is defined, first for $f \in \mathcal{F} \cap L^\infty(X,m)$ as the unique ($[0,\infty]$-valued) Borel measure on X such that

$$\int_X g\, d\Gamma(f,f) = \mathcal{E}(f, fg) - \frac{1}{2}\mathcal{E}(f^2, g) \qquad \text{for all } g \in \mathcal{F} \cap C_\mathrm{c}(X), \tag{2.1}$$

and then by $\Gamma(f, f)(A) := \lim_{n\to\infty} \Gamma\big((-n) \vee (f \wedge n), (-n) \vee (f \wedge n)\big)(A)$ for each Borel subset A of X for general $f \in \mathcal{F}$.

Definition 2.2 ([46, Definition 2.1]) Let $(X, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space. A σ-finite Borel measure ν on X is called a *minimal energy-dominant measure* of $(\mathcal{E}, \mathcal{F})$ if the following two conditions are satisfied:

(i) (Domination) For every $f \in \mathcal{F}$, $\Gamma(f, f) \ll \nu$.
(ii) (Minimality) If another σ-finite Borel measure ν' on X satisfies condition (i) with ν replaced by ν', then $\nu \ll \nu'$.

Note that by [46, Lemmas 2.2, 2.3 and 2.4], a minimal energy-dominant measure of $(\mathcal{E}, \mathcal{F})$ always exists and is precisely a σ-finite Borel measure ν on X such that for each Borel subset A of X, $\nu(A) = 0$ if and only if $\Gamma(f, f)(A) = 0$ for all $f \in \mathcal{F}$. In particular, any two minimal energy-dominant measures are mutually absolutely continuous.

Definition 2.3 Let $(X, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space. We define its *intrinsic metric* $d_{\mathrm{int}}\colon X \times X \to [0, \infty]$ by

$$d_{\mathrm{int}}(x, y) := \sup\{f(x) - f(y) \mid f \in \mathcal{F}_{\mathrm{loc}} \cap C(X), \Gamma(f, f) \le m\}, \quad (2.2)$$

where

$$\mathcal{F}_{\mathrm{loc}} := \left\{ f \;\middle|\; \begin{array}{l} f \text{ is an } m\text{-equivalence class of } \mathbb{R}\text{-valued Borel measurable} \\ \text{functions on } X \text{ such that } f\mathbb{1}_V = f^{\#}\mathbb{1}_V \ m\text{-a.e. for some} \\ f^{\#} \in \mathcal{F} \text{ for each relatively compact open subset } V \text{ of } X \end{array} \right\} \quad (2.3)$$

and the energy measure $\Gamma(f, f)$ of $f \in \mathcal{F}_{\mathrm{loc}}$ associated with $(X, d, m, \mathcal{E}, \mathcal{F})$ is defined as the unique Borel measure on X such that $\Gamma(f, f)(A) = \Gamma(f^{\#}, f^{\#})(A)$ for any relatively compact Borel subset A of X and any V, $f^{\#}$ as in (2.3) with $A \subset V$; note that $\Gamma(f^{\#}, f^{\#})(A)$ is independent of a particular choice of such V, $f^{\#}$ by [33, Corollary 3.2.1].

We remark that the intrinsic metric need not always be a metric. In general, it is only a pseudo-metric; that is, $d_{\mathrm{int}}(x, x) = 0$ for all $x \in X$, $d_{\mathrm{int}}(x, y) \in [0, \infty]$, $d_{\mathrm{int}}(x, y) = d_{\mathrm{int}}(y, x)$ for all $x, y \in X$ and $d_{\mathrm{int}}(x, z) \le d_{\mathrm{int}}(x, y) + d_{\mathrm{int}}(y, z)$ for all $x, y, z \in X$. A sufficient condition for when the intrinsic metric is bi-Lipschitz equivalent to the original metric is given in Proposition 2.11. On the other hand, [56, Theorem 2.13-(a)] gives a family of examples for which the intrinsic metric is identically zero. In the setting of [56, Theorem 2.13-(a)], $\Gamma(f, f) \le m$ implies that f is identically constant. We refer the reader to [13, Propostion 5.7] for an example which shows that the intrinsic metric could be infinite.

2.2 Harnack inequalities

We recall the definition of harmonic and caloric functions.

Definition 2.4 Let $(X, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space and let $\mathcal{F}_e$ denote its extended Dirichlet space. Recall that the *extended Dirichlet space* $\mathcal{F}_e$ of $(X, d, m, \mathcal{E}, \mathcal{F})$ is defined as the space of m-equivalence classes of functions $f : X \to \mathbb{R}$ such that $\lim_{n\to\infty} f_n = f$ m-a.e. on X for some $\mathcal{E}$-Cauchy sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{F}$, that the limit $\mathcal{E}(f, f) := \lim_{n\to\infty} \mathcal{E}(f_n, f_n) \in \mathbb{R}$ exists, is independent of a choice of such $(f_n)_{n\in\mathbb{N}}$ for each $f \in \mathcal{F}_e$ and defines an extension of $\mathcal{E}$ to $\mathcal{F}_e \times \mathcal{F}_e$, and that $\mathcal{F} = \mathcal{F}_e \cap L^2(X, m)$; see [24, Definition 1.1.4 and Theorem 1.1.5]. We remark that the definition of the energy measure $\Gamma(f, f)$ associated with $(X, d, m, \mathcal{E}, \mathcal{F})$ also extends canonically to $f \in \mathcal{F}_e$; see [33, p. 123 and Theorem 5.2.3].

A function $h \in \mathcal{F}_e$ is said to be $\mathcal{E}$*-harmonic* on an open subset U of X, if

$$\mathcal{E}(h, f) = 0 \qquad \text{for all } f \in \mathcal{F} \cap C_c(X) \text{ with } \operatorname{supp}_m[f] \subset U. \tag{2.4}$$

Let I be an open interval in $\mathbb{R}$. We say that a function $u : I \to L^2(X, m)$ is *weakly differentiable* at $t_0 \in I$ if for any $f \in L^2(X, m)$ the function $t \mapsto \langle u(t), f\rangle$ is differentiable at t_0, where $\langle\cdot,\cdot\rangle$ denotes the inner product in $L^2(X, m)$. If u is weakly differentiable at t_0, then by the uniform boundedness principle, there exists a (unique) function $w \in L^2(X, m)$ such that

$$\lim_{t\to t_0} \left\langle \frac{u(t) - u(t_0)}{t - t_0}, f \right\rangle = \langle w, f\rangle, \qquad \text{for all } f \in L^2(X, m).$$

We say that the function w is the *weak derivative* of the function u at t_0 and write $w = u'(t_0)$.

Let I be an open interval in $\mathbb{R}$ and let Ω be an open subset of X. A function $u : I \to \mathcal{F}$ is said to be *caloric* in $I \times \Omega$ if u is weakly differentiable in the space $L^2(\Omega)$ at any $t \in I$, and for any $f \in \mathcal{F} \cap C_c(\Omega)$, and for any $t \in I$,

$$\langle u', f\rangle + \mathcal{E}(u, f) = 0. \tag{2.5}$$

Definition 2.5 (Harnack inequalities) We say that an MMD space $(X, d, m, \mathcal{E}, \mathcal{F})$ satisfies the *elliptic Harnack inequality* (abbreviated as EHI), if there exist $C > 1$ and $\delta \in (0, 1)$ such that for all $x \in X$, $r > 0$ and for any $h \in \mathcal{F}_e$ that is non-negative on $B(x, r)$ and $\mathcal{E}$-harmonic on $B(x, r)$, we have

$$\operatorname*{ess\,sup}_{B(x,\delta r)} h \le C \operatorname*{ess\,inf}_{B(x,\delta r)} h. \tag{EHI}$$

We say that an MMD space $(X, d, m, \mathcal{E}, \mathcal{F})$ satisfies the *parabolic Harnack inequality* with walk dimension β (abbreviated as PHI(β)), if there exist $0 < C_1 < C_2 < C_3 < C_4 < \infty$, $C_5 > 1$ and $\delta \in (0, 1)$ such that for all $x \in X$, $r > 0$ and for any non-negative bounded caloric function u on the space-time cylinder $Q = (a, a + C_4 r^\beta) \times B(x, r)$, we have

$$\operatorname*{ess\,sup}_{Q_-} u \le C_5 \operatorname*{ess\,inf}_{Q_+} u, \tag{PHI(β)}$$

where $Q_- = (a + C_1 r^\beta, a + C_2 r^\beta) \times B(x, \delta r)$ and $Q_+ = (a + C_3 r^\beta, a + C_4 r^\beta) \times B(x, \delta r)$.

Remark 2.6 The formulation of EHI in Definition 2.5 above is slightly different from that in [9, Definition 4.2-(i)], but it is easy to see from the relative compactness in X of all balls in (X, d) and [9, Proposition 2.9-(ii)] that the former implies the latter, so that we can freely use the implications of EHI established in [9] under the present formulation of EHI.

2.3 Admissible measures and time-changed Dirichlet space

Given an MMD space $(X, d, m, \mathcal{E}, \mathcal{F})$ and $A \subset X$, we define its 1-capacity as

$$\mathrm{Cap}_1(A) := \inf\bigl\{\mathcal{E}_1(f, f) \bigm| f \in \mathcal{F},\ f \ge 1\ m\text{-a.e. on a neighborhood of } A\bigr\}, \tag{2.6}$$

where $\mathcal{E}_1 := \mathcal{E} + \langle \cdot, \cdot \rangle_{L^2(X,m)}$ as defined before. For disjoint Borel sets A, B such that B is closed and $\overline{A} \Subset B^c$ (by $A \Subset B^c$, we mean that $\overline{A}$ is compact and $\overline{A} \subset B^c$), we define $\mathcal{F}(A, B)$ as the set of function $\phi \in \mathcal{F}$ such that $\phi \equiv 1$ in an open neighborhood of A, and $\operatorname{supp} \phi \subset B^c$. For such sets A and B, we define the capacity between them as

$$\mathrm{Cap}(A, B) := \inf \{\mathcal{E}(f, f) \mid f \in \mathcal{F}(A, B)\}. \tag{2.7}$$

Definition 2.7 (Smooth measures) Let $(X, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space. A Radon measure μ on X, i.e., a Borel measure μ on X which is finite on any compact subset of X, is said to be *smooth* if μ charges no set of zero capacity (that is, $\mu(A) = 0$ for any Borel subset A of X with $\mathrm{Cap}_1(A) = 0$).

For example, the energy measure $\Gamma(f, f)$ of $f \in \mathcal{F}_e$ is smooth by [33, Lemma 3.2.4]. An essential feature of a smooth Radon measure μ on X is that the μ-equivalence class of each $f \in \mathcal{F}_e$ is canonically determined by considering a *quasi-continuous* m-version of f, which exists by [33, Theorem

2.1.7] and is unique *q.e.* (i.e., up to sets of capacity zero) by [33, Lemma 2.1.4]; see [33, Section 2.1] and [24, Sections 1.2, 1.3 and 2.3] for the definition and basic properties of quasi-continuous functions with respect to a regular symmetric Dirichlet form. In what follows, we always consider a quasi-continuous m-version of $f \in \mathcal{F}_e$.

An increasing sequence $\{F_k; k \geq 1\}$ of closed subsets of an MMD space $(X, d, m, \mathcal{E}, \mathcal{F})$ is said to be a *nest* if $\bigcup_{k\geq 1} \mathcal{F}_{F_k}$ is $\sqrt{\mathcal{E}_1}$-dense in $\mathcal{F}$, where $\mathcal{F}_{F_k} := \{f \in \mathcal{F} \mid f = 0\ m\text{-a.e. on } X \setminus F_k\}$. Recall that $D \subset X$ is *quasi-open* if there exists a nest $\{F_n\}$ such that $D \cap F_n$ is an open subset of F_n in the relative topology for each $n \in \mathbb{N}$. The complement in X of a quasi-open set is called *quasi-closed*. We are interested in the class of admissible measures, namely smooth Radon measures having full quasi-support in the sense defined as follows.

Definition 2.8 (Full quasi-support, admissible measures; [33, (4.6.3) and (4.6.4)]) Let $(X, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space. A smooth Radon measure μ on X is said to have *full quasi-support* if for any quasi-closed set F with $\mu(X \setminus F) = 0$ we have $\mathrm{Cap}_1(X \setminus F) = 0$. A Borel measure μ on X is said to be *admissible* if μ is a smooth Radon measure on X with full quasi-support, and the set of admissible measures with respect to $(X, d, m, \mathcal{E}, \mathcal{F})$ is denoted by $\mathcal{A}(X, d, m, \mathcal{E}, \mathcal{F})$.

Definition 2.9 (Time-changed Dirichlet form) Let $(X, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space. If μ is a smooth Radon measure, it defines a time change of the process whose associated Dirichlet form is called the trace Dirichlet form and denoted by $(\mathcal{E}^\mu, \mathcal{F}^\mu)$ (see [33, Section 6.2] and [24, Section 5.2]). Assume $\mu \in \mathcal{A}(X, d, m, \mathcal{E}, \mathcal{F})$. Then the trace Dirichlet form $(\mathcal{E}^\mu, \mathcal{F}^\mu)$ is given by

$$\mathcal{F}^\mu = \mathcal{F}_e \cap L^2(X, \mu) \qquad \text{and} \qquad \mathcal{E}^\mu(u, v) = \mathcal{E}(u, v) \quad \text{for } u, v \in \mathcal{F}^\mu, \tag{2.8}$$

and $(\mathcal{E}^\mu, \mathcal{F}^\mu)$ is a strongly local, regular symmetric Dirichlet form on $L^2(X, \mu)$ by [33, Theorems 5.1.5, 6.2.1 and Exercise 3.1.1]. We also note that by [24, Theorem 5.2.11],

$$\mathcal{A}(X, d, m, \mathcal{E}, \mathcal{F}) = \mathcal{A}(X, d, \mu, \mathcal{E}^\mu, \mathcal{F}^\mu). \tag{2.9}$$

In probabilistic terms, $(\mathcal{E}^\mu, \mathcal{F}^\mu)$ is the Dirichlet form of the time-changed process $(\omega, t) \mapsto Y_{\tau_t(\omega)}(\omega)$, where $(Y_t)_{t\geq 0}$ is an m-symmetric diffusion on X whose Dirichlet form is $(\mathcal{E}, \mathcal{F})$ and τ_t is the right-continuous inverse of the positive continuous additive functional $(A_t)_{t\geq 0}$ of $(Y_t)_{t\geq 0}$ with Revuz measure μ; see [33, Section 6.2] and [24, Section 5.2] for details.

2.4 Main results

Our first main result is that the value of the conformal walk dimension is an invariant for MMD spaces satisfying the metric doubling property and the elliptic Harnack inequality EHI. We recall from Definition 1.2 that the conformal walk dimension d_{cw} of an MMD space $(X, d, m, \mathcal{E}, \mathcal{F})$ is the infimum over all $\beta > 0$ such that there exist an admissible measure $\mu \in \mathcal{A}(X, d, m, \mathcal{E}, \mathcal{F})$ and a metric $\theta \in \mathcal{J}(X, d)$ such that the time-changed MMD space $(X, \theta, \mu, \mathcal{E}^{\mu}, \mathcal{F}^{\mu})$ satisfies the parabolic Harnack inequality PHI(β); note here that $(X, \theta, \mu, \mathcal{E}^{\mu}, \mathcal{F}^{\mu})$ is indeed an MMD space since, for any $(x, r) \in X \times (0, \infty)$, $\mathrm{diam}(B_{\theta}(x, r), d) < \infty$ by $\theta \in \mathcal{J}(X, d)$ and [40, Proposition 10.8] and hence $B_{\theta}(x, r)$ is relatively compact in X.

Theorem 2.10 (Universality of conformal walk dimension) *Let $(X, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space and let d_{cw} denote its conformal walk dimension. Then the following are equivalent:*

(a) $(X, d, m, \mathcal{E}, \mathcal{F})$ satisfies the metric doubling property and EHI .
(b) $d_{\mathrm{cw}} < \infty$.
(c) $d_{\mathrm{cw}} = 2$.

The proof of Theorem 2.10 is concluded in Sect. 4.2 after long preparations in Sects. 3 and 4.1. We give a brief description of the proof in Sect. 2.5.

The next question is whether or not the infimum in the definition (1.4) of d_{cw} is attained. To this end we first describe the metric and measure. The following result is essentially contained in [56]; see the beginning of Sect. 5.1 for the proof.

Proposition 2.11 *Let $(X, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space that satisfies* PHI(2). *Then the following hold:*

(a) The metric d is bi-Lipschitz equivalent to the intrinsic metric d_{int}.
(b) The symmetric measure m is a minimal energy-dominant measure.

By Proposition 2.11-(a), in order to find a metric $\theta \in \mathcal{J}(X, d)$ and a measure $\mu \in \mathcal{A}(X, d, m, \mathcal{E}, \mathcal{F})$ in the Gaussian uniformization problem, it is enough to find an appropriate measure μ as the metric θ is determined by the measure up to bi-Lipschitz equivalence. This observation is useful in studying the Gaussian uniformization problem, since constructing measures is typically easier than constructing metrics. By Proposition 2.11-(b), any such measures are mutually absolutely continuous. In fact, we have the following improvement, asserting that any two measures μ_1, μ_2 attaining the infimum in the definition of d_{cw} are A_{∞}-related in the sense of Muckenhoupt (see Definition 5.5).

Theorem 2.12 *Let $(X, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space. For $i = 1, 2$, let $d_i \in \mathcal{J}(X, d)$, $m_i \in \mathcal{A}(X, d, m, \mathcal{E}, \mathcal{F})$ and assume that the time-changed MMD* *space* $(X, d_i, m_i, \mathcal{E}^{m_i}, \mathcal{F}^{m_i})$ *satisfies* PHI(2). *Then the measures* m_1 *and* m_2 *are* A_∞*-related in* (X, d).

The proof of Theorem 2.12 is given in the first half of Sect. 5.2. Using Proposition 2.11, Theorem 2.12 and sharp constants of Poincaré inequalities in [26], we also answer in Theorem 5.18 the Gaussian uniformization problem for the MMD space $(\mathbb{R}, d, m, \mathcal{E}, \mathcal{F})$ corresponding to the Brownian motion on $\mathbb{R}$, by proving that any Borel measure μ on $\mathbb{R}$ that is A_∞-related to m admits $\theta \in \mathcal{J}(\mathbb{R}, d)$ such that $(\mathbb{R}, \theta, \mu, \mathcal{E}^\mu, \mathcal{F}^\mu)$ satisfies PHI(2). This result is not true for the MMD space $(\mathbb{R}^n, d, m, \mathcal{E}, \mathcal{F})$ corresponding to the Brownian motion on $\mathbb{R}^n$ with $n \geq 2$ as shown in Example 5.14, and we do not have an exact characterization of the Borel measures on $\mathbb{R}^n$ which are A_∞-related to m and admit $\theta \in \mathcal{J}(\mathbb{R}^n, d)$ such that $(\mathbb{R}^n, \theta, \mu, \mathcal{E}^\mu, \mathcal{F}^\mu)$ satisfies PHI(2), except that we give an implicit one for $n = 2$ in Proposition 5.16.

2.5 Outline of the proof of Theorem 2.10

As pointed out in the introduction, the equivalence between (a) and (b) is already known. Since the implication from (c) to (b) is trivial, it suffices to show that (b) implies (c). Hence, we may assume that $(X, d, m, \mathcal{E}, \mathcal{F})$ satisfies PHI(γ) for some $\gamma > 2$. Let $\beta > 2$ be arbitrary. We wish to construct a metric $\theta \in \mathcal{J}(X, d)$ and $\mu \in \mathcal{A}(X, d, m, \mathcal{E}, \mathcal{F})$ such that the time-changed MMD space $(X, \theta, \mu, \mathcal{E}^\mu, \mathcal{F}^\mu)$ satisfies PHI(β).

To sketch the main ideas, we further assume that (X, d) is compact and the diameter of (X, d) is normalized to $\frac{1}{3}$. The non-compact case follows by the same argument as the compact case, by considering X as a limit of compact sets. Thanks to the known characterizations of PHI(β) as stated in Theorem 4.5 and the preservation of EHI between $(X, d, m, \mathcal{E}, \mathcal{F})$ and $(X, \theta, \mu, \mathcal{E}^\mu, \mathcal{F}^\mu)$ as stated in Lemma 4.8, it is enough to construct a metric $\theta \in \mathcal{J}(X, d)$ and a measure $\mu \in \mathcal{A}(X, d, m, \mathcal{E}, \mathcal{F})$ such that

$$\mu(B_\theta(x, r)) \asymp r^\beta \operatorname{Cap}(B_\theta(x, r), B_\theta(x, 2r)^c), \quad \text{for all } x \in X, r \lesssim \operatorname{diam}(X, \theta). \tag{2.10}$$

The above estimate relating the measure μ and capacity implies that μ is a smooth measure with full quasi-support and satisfies the following volume doubling and reverse volume doubling properties (see Proposition 4.14): there exists $C_D > 0$ and $c_D \in (0, 1)$ such that

$$\frac{\mu(B_d(x,2r))}{\mu(B_d(x,r))} \le C_D, \quad \text{for all } x \in X, r > 0, \text{ and} \tag{2.11}$$
$$\frac{\mu(B_d(x,r))}{\mu(B_d(x,2r))} \le c_D, \quad \text{for all } x \in X, 0 < r \lesssim \operatorname{diam}(X,d). \tag{2.12}$$

The estimate (2.10) along with Theorem 4.5, implies PHI(β) because volume doubling, reverse volume doubling, and EHI are preserved by the quasisymmetric change of the metric from d to θ.

The construction of a metric θ and a measure μ is a modification of [23], but instead of the Ahlfors regularity required in [23] we need to establish (2.10). Following [23], we construct the metric θ and the measure μ that satisfy (2.10) using a multi-scale argument. This part of the argument relies on theory of Gromov hyperbolic spaces. The basic idea behind the approach is to construct a Gromov hyperbolic graph (called the *hyperbolic filling*) whose boundary (in the sense of Gromov) corresponds to the given metric space (X, d). A well-known result in Gromov hyperbolic spaces asserts that any metric in the conformal gauge $\mathcal{J}(X, d)$ up to bi-Lipschitz equivalence can be obtained by a bounded perturbation of edge weights on the hyperbolic filling. We recall the basic results about Gromov hyperbolic spaces and their boundaries in Sect. 3.1.

We first sketch the construction of this hyperbolic graph, postponing a more precise definition to Sect. 3.2. We choose a parameter $a > 10^2$ and cover the space X using a covering $\mathcal{S}_n$ with balls of radii $2a^{-n}$ such that for any two distinct balls $B_d(x_1, 2a^{-n}), B_d(x_2, 2a^{-n}) \in \mathcal{S}_n$, we have $B_d(x_1, a^{-n}/2) \cap B_d(x_2, a^{-n}/2) = \emptyset$ (we think of these balls as 'approximately pairwise disjoint covering' at scale a^{-n}). Therefore the covering $\mathcal{S}_n$ corresponds to scale a^{-n} for all $n \in \mathbb{N}_{\ge 0}$. In what follows, for a ball B, we denote by x_B and r_B the center and radius of B. For a ball B and $\lambda > 0$, we denote by $\lambda \cdot B$, the ball $B_d(x_B, \lambda r_B)$.

We define a *tree of vertical edges* with vertex set $\coprod_{n\ge 0} \mathcal{S}_n$ by choosing for each ball $B \in \mathcal{S}_n, n \ge 1$ a 'parent ball' $B' \in \mathcal{S}_{n-1}$ such that $x_{B'}$ is a closest point to x_B in the set $\{x_C : C \in \mathcal{S}_{n-1}\}$. By the assumption on the diameter, $\mathcal{S}_0$ is a singleton. The edges in this tree are called *vertical edges*. We choose another parameter $\lambda \ge 10$ to define another set of edges on $\coprod_{n\ge 0} \mathcal{S}_n$ called *horizontal edges*. Two distinct balls $B, \widetilde{B} \in \mathcal{S}_n, n \ge 0$ share a horizontal edge if and only if $\lambda \cdot B \cap \lambda \cdot \widetilde{B} \ne \emptyset$. The set of edges of the hyperbolic filling is defined as the union of the sets of horizontal and vertical edges.

In our construction, the vertical edge weights play a more central role and the values of horizontal edge weights are less important. The weight of the vertical edge between $B \in \mathcal{S}_n, B' \in \mathcal{S}_{n-1}$ can be interpreted as the relative diameter under the θ-metric. To describe the data required in our construction of the metric, suppose for the moment that θ is a metric on X with the desired properties, and let us define the *relative diameter* of $B \in \mathcal{S}_n, n \geq 1$ as

$$\rho(B) := \frac{\mathrm{diam}(B, \theta)}{\mathrm{diam}(B', \theta)}, \tag{2.13}$$

where $B' \in \mathcal{S}_{n-1}$ is such that there is a vertical edge between B' and B (B' is the parent of B in the tree of vertical edges). It turns out that the 'relative diameter' in (2.13) contains enough information about θ to reconstruct the metric θ (up to bi-Lipschitz equivalence). Hence we could reduce the problem of construction of $\theta \in \mathcal{J}(X, d)$ to constructing the function $\rho(\cdot)$ on $\coprod_{n\geq 1} \mathcal{S}_n$; see Theorem 3.14.

It is therefore enough to construct $\rho(\cdot)$ in (2.13). Next, we describe two key conditions that the relative diameter $\rho(\cdot)$ defined in (2.13) must satisfy. For a ball $B \in \mathcal{S}_{n-1}, n \geq 1$, let us denote by $\Gamma_n(B)$ the set of horizontal paths in $\mathcal{S}_n$ defined by

$$\Gamma_n(B) = \left\{ (B_i)_{i=0}^N \;\middle|\; \begin{array}{l} N \in \mathbb{N},\, B_i \in \mathcal{S}_n \text{ for all } i = 0, \dots, N;\, B_i \text{ and } B_{i+1} \\ \text{share a horizontal edge for all } i = 0, \dots, N-1, \\ x_{B_0} \in B,\, x_{B_1}, \dots, x_{B_{N-1}} \in 2 \cdot B,\, x_{B_N} \notin 2 \cdot B \end{array} \right\}.$$

The first condition on $\rho(\cdot)$ is

$$\sum_{i=1}^N \rho(B_i) \gtrsim 1, \quad \text{for all } (B_i)_{i=0}^N \in \Gamma_n(B) \text{ and } B \in \mathcal{S}_{n-1}, n \geq 1. \tag{2.14}$$

The condition (2.14) is a consequence of the fact that $\theta \in \mathcal{J}(X, d)$ and that (X, d) is a uniformly perfect metric space. We like to think of (2.14) as a '*no shortcuts condition*' as it disallows the possibility of short cuts in the θ-metric from $B_d(x, r)$ to $B_d(x, 2r)^c$.

The second condition arises from the estimate (2.10). For any ball $B \in \mathcal{S}_k, k \geq 0$, by $\mathcal{D}_{k+1}(B)$, we denote its descendants in $\mathcal{S}_{k+1}$; that is $\mathcal{D}_{k+1}(B)$ is the set of elements $B' \in \mathcal{S}_{k+1}$ such that B' and B share a vertical edge. The second condition that ρ must satisfy is the following estimate:

$$\sum_{B' \in \mathcal{D}_{k+1}(B)} \rho(B')^\beta \,\mathrm{Cap}(B', (2 \cdot B')^c) \lesssim \mathrm{Cap}(B, (2 \cdot B)^c), \tag{2.15}$$

for all $B \in \mathcal{S}_k$ and k large enough. To explain that (2.15) should hold for a metric θ and a measure μ with the desired properties, we first observe that the volume doubling property (2.11) implies that

$$\sum_{B' \in \mathcal{D}_{k+1}(B)} \mu(B') \lesssim \mu(B), \quad \text{for all } B \in \mathcal{S}_k.$$

By (2.10), $\theta \in \mathcal{J}(X, d)$ and comparing capacity of annuli in θ and d metrics on the basis of EHI [9, Lemmas 5.22 and 5.23], one obtains $\mu(B) \asymp \operatorname{diam}(B, \theta)^\beta \operatorname{Cap}(B, (2 \cdot B)^c)$ for all $B \in \mathcal{S}_k$ and for all large enough k. Combining these estimates with (2.11), we obtain (2.15). To summarize, the conditions (2.14) and (2.15) arise from the metric and the measure, respectively. It turns out that the necessary conditions (2.14) and (2.15) on $\rho(\cdot)$ are 'almost sufficient' to construct ρ; see Theorem 3.24.

We note that there is a tension between the estimates (2.14) and (2.15). On the one hand, in order to satisfy (2.14), the function $\rho(\cdot)$ must be large enough, whereas (2.15) imposes that $\rho(\cdot)$ cannot be too large. Next, we sketch how to construct ρ that satisfies these seemingly conflicting requirements in (2.14) and (2.15). Let $B \in \mathcal{S}_k$, $u \in C_c(X)$ be such that

$$u \equiv 1 \text{ on } B_d(x_B, 1.1 r_B), \quad u \equiv 0 \text{ on } B_d(x_B, 1.9 r_B)^c, \quad \mathcal{E}(u, u) \asymp \frac{m(B)}{r_B^\gamma}. \tag{2.16}$$

Such a function exists because of (2.10) for $(X, d, m, \mathcal{E}, \mathcal{F})$ with β replaced by γ and a covering argument (the constants 1.1 and 1.9 are essentially arbitrary and could be replaced by $1 + \varepsilon$ and $2 - \varepsilon$ for small enough ε). It is helpful to think of u as the equilibrium potential corresponding to $\operatorname{Cap}(B_d(x_B, 1.1 r_B), B_d(x_B, 1.9 r_B)^c)$. Let us define the functions $u_B, \rho_B : \mathcal{S}_{k+1} \to [0, \infty)$ as

$$u_B(B') := \fint_{B'} u\, dm = \frac{1}{m(B')} \int_{B'} u\, dm,$$
$$\rho_B(B'') = \sum_{B'' \in \mathcal{S}_{k+1}, B'' \sim B'} \left| u_B(B'') - u_B(B') \right|,$$

where $B'' \sim B'$ means that B'' and B' share a horizontal edge (or equivalently, $\lambda \cdot B'' \cap \lambda \cdot B' \neq \emptyset$). From the definitions it is clear that u_B is a discrete version of u and ρ_B is a discrete version of the gradient of u. Using a Poincaré inequality on $(X, d, m, \mathcal{E}, \mathcal{F})$ and the bound $\operatorname{Cap}(B', (2 \cdot B')^c) \asymp \frac{m(B')}{r_{B'}^\gamma}$, we obtain the following estimate (see (4.22)):

$$\sum_{B' \in \mathcal{D}_{k+1}(B)} \rho_B(B')^2 \operatorname{Cap}(B', (2 \cdot B')^c) \lesssim \mathcal{E}(u, u) \lesssim \operatorname{Cap}(B, (2 \cdot B)^c). \tag{2.17}$$

Since $u_B(B_0) = 1$ for any $B_0 \in \mathcal{S}_{k+1}$ with $x_{B_0} \in B$ and $u_B(B_N) = 0$ for any $B_N \in \mathcal{S}_{k+1}$ with $x_{B_N} \notin (2 \cdot B)^c$, by the triangle inequality

$$\sum_{i=1}^{N} \rho_B(B_i) \geq 1, \quad \text{for all } (B_i)_{i=0}^{N} \in \Gamma_{k+1}(B). \tag{2.18}$$

Clearly, (2.18) and (2.17) are local versions of (2.14) and (2.15) with $\beta = 2$, respectively. Here, by "local version" we mean that the estimates (2.14) and (2.15) are satisfied for a function ρ_B dependent on B for each fixed B. To ensure (2.14) and (2.15) for all scales and locations, we define

$$\rho(B') = \sup_{B \in \mathcal{S}_k} \rho_B(B'), \quad \text{for all } B' \in \mathcal{S}_{k+1}, \tag{2.19}$$

where ρ_B is defined as above at all locations and scales. This ensures (2.14) and (2.15) at all scales with $\beta = 2$.

However, ρ should satisfy further additional conditions that the above construction need not obey. Since $\theta \in \mathcal{J}(X, d)$, one obtains that

$$\rho(B) \gtrsim 1 \quad \text{for all } B \in \mathcal{S}_k, k \geq 1. \tag{2.20}$$

However, the function $\rho(\cdot)$ defined in (2.19) need not satisfy the estimate (2.20). This requires us to increase ρ further if necessary. We define the 'diameter function'

$$\pi(B) = \prod_{k=0}^{n} \rho(B_i), \quad \text{for all } B \in \mathcal{S}_n, \tag{2.21}$$

where $B_i \in \mathcal{S}_i$ for all $i = 0, \ldots, n$, $B_n = B$ and there is a vertical edge between B_i and B_{i+1} for all $i = 0, \ldots, n-1$. If ρ is given by (2.13), then clearly $\pi(B) = \operatorname{diam}(B, \theta) / \operatorname{diam}(X, \theta)$. By quasisymmetry, $\operatorname{diam}(B, \theta) \asymp \operatorname{diam}(B', \theta)$ whenever B and B' share a horizontal edge. This suggests the following condition on ρ:

$$\pi(B) \asymp \pi(B'), \quad \text{whenever } B \text{ and } B' \text{ share a horizontal edge}, \tag{2.22}$$

where π is defined as given in (2.21). Similarly, for constructing measure, we need to ensure that ρ satisfies

$$\pi(B)^{\beta} \operatorname{Cap}(B, (2 \cdot B)^c) \asymp \sum_{B' \in \mathcal{D}_n(B)} \pi(B')^{\beta} \operatorname{Cap}(B', (2 \cdot B')^c), \tag{2.23}$$

for all $B \in \mathcal{S}_k$ and $n > k \geq 0$, where $\mathcal{D}_n(B)$ denotes the descendants of B in $\mathcal{S}_n$. The conditions (2.22) and (2.23) are rather delicate because the value of π can change drastically if we change ρ by a bounded multiplicative factor. This is due to the fact that the multiplicative 'errors' in ρ accumulate as we move to finer and finer scales. This suggests that we need to control the constants very carefully.

To achieve this we need to consider $\beta > 2$ (instead of $\beta = 2$ considered above). We choose ρ defined in (2.19) uniformly small by picking a function u that satisfies (2.16) along with an additional scale invariant Hölder continuity estimate (see (4.27), (4.28) and (4.29)). Then using

$$\begin{aligned}&\sum_{B'\in\mathcal{D}_{k+1}(B)} \rho_B(B')^\beta \operatorname{Cap}(B', (2\cdot B')^c)\\ &\leq \|\rho\|_\infty^{\beta-2} \sum_{B'\in\mathcal{D}_{k+1}(B)} \rho_B(B')^2 \operatorname{Cap}(B', (2\cdot B')^c) \lesssim \|\rho\|_\infty^{\beta-2} \operatorname{Cap}(B, (2\cdot B)^c),\end{aligned}$$

we obtain enough control on the constants in (2.15) to ensure (2.20) and (2.22) after further modification of ρ. By the Hölder continuity estimate on u, $\|\rho\|_\infty$ can be made arbitrarily small by increasing the vertical parameter a.

3 Metric and measure via hyperbolic filling

Given a metric space, it is often useful to view the space as the boundary of a Gromov hyperbolic space. Such a viewpoint is prevalent but often implicit in various multi-scale arguments in analysis and probability. Roughly speaking, a metric space viewed simultaneously at different locations and scales has a natural hyperbolic structure. A nice introduction to this viewpoint can be found in [84]. This will be made precise by the notion of hyperbolic filling in Sect. 3.2. The main tool for the construction of metric is Theorem 3.14-(a), and the construction of measure uses Lemma 3.20. To describe the construction, we recall the definition of hyperbolic space in the sense of Gromov.

3.1 Gromov hyperbolic spaces and their boundary

We briefly recall the basics of Gromov hyperbolic spaces and refer the reader to [27,34,39,91] for a detailed exposition.

Let (Z, d) be a metric space. Given three points $x, y, p \in Z$, we define the *Gromov product* of x and y with respect to the base point w as

$$(x|y)_w = \frac{1}{2}(d(x,w) + d(y,w) - d(x,y)). \tag{3.1}$$

By the triangle inequality, Gromov product is always non-negative. We say that a metric space (Z, d) is *δ-hyperbolic*, if for any four points $x, y, z, w \in Z$, we have

$$(x|z)_w \geq (x|y)_w \wedge (y|z)_w - \delta.$$

We say that (Z, d) is hyperbolic (or d is a hyperbolic metric), if (Z, d) is hyperbolic for some $\delta \in [0, \infty)$. If the above condition is satisfied for a fixed base point $w \in Z$, and arbitrary $x, y, z \in Z$, then (Z, d) is 2δ-hyperbolic [27, Proposition 1.2].

Next, we recall the notion of the boundary of a hyperbolic space. Let (Z, d) be a hyperbolic space and let $w \in Z$. A sequence of points $\{x_i\} \subset Z$ is said to *converge at infinity*, if

$$\lim_{i,j \to \infty} (x_i|x_j)_w = \infty.$$

The above notion of convergence at infinity does not depend on the choice of the base point $w \in Z$, because by the triangle inequality $|(x|y)_w - (x|y)_{w'}| \leq d(w, w')$.

Two sequences $\{x_i\}, \{y_i\}$ that converge at infinity are said to be *equivalent*, if

$$\lim_{i \to \infty} (x_i|y_i)_w = \infty.$$

This defines an equivalence relation among all sequences that converge at infinity. As before, is easy to check that the notion of equivalent sequences does not depend on the choice of the base point w. The *boundary* ∂Z of (Z, d) is defined as the set of equivalence classes of sequences converging at infinity under the above equivalence relation. If there are multiple hyperbolic metrics on the same set Z, to avoid confusion, we denote the boundary of (Z, d) by $\partial(Z, d)$ (see Lemma 3.13-(d)). The notion of Gromov product can be defined on the boundary as follows: for all $a, b \in \partial Z$

$$(a|b)_w = \sup \left\{ \liminf_{i \to \infty} (x_i|y_i)_w : \{x_i\} \in a, \{y_i\} \in b \right\},$$

and similarly, for $a \in \partial Z$, $y \in Z$, we define

$$(a|y)_w = \sup \left\{ \liminf_{i \to \infty} (x_i|y)_w : \{x_i\} \in a \right\}.$$

The boundary ∂Z of the hyperbolic space (Z, d) carries a family of metrics. Let $w \in Z$ be a base point. A metric $\rho : \partial Z \times \partial Z \to [0, \infty)$ on ∂Z is said to be a *visual metric* with *visual parameter* $\alpha > 1$ if there exists $k_1, k_2 > 0$ such that

$$k_1 \alpha^{-(a|b)_w} \leq \rho(a,b) \leq k_2 \alpha^{-(a|b)_w}.$$

Note that visual metrics depend on the choice of the base point, and on the visual parameter α. If a visual metric with base point w and visual parameter α exists, then it can be chosen to be

$$\rho_{\alpha,w}(a,b) := \inf \sum_{i=1}^{n-1} \alpha^{-(a_i|a_{i+1})_w},$$

where the infimum is over all finite sequences $\{a_i\}_{i=1}^n \subset \partial Z, n \geq 2$ such that $a_1 = a, a_n = b$.

Visual metrics exist as we recall now. A metric space (Z,d) is said to be *proper* if all closed balls are compact. For any δ-hyperbolic space (Z,d), there exists $\alpha_0 > 1$ (α depends only on δ) such that if $\alpha \in (1, \alpha_0)$, then there exists a visual metric with parameter α [34, Chapitre 7], [19, Lemma 6.1]. It is well known that quasi-isometry between almost geodesic hyperbolic spaces induces a quasisymmetry on their boundaries (the notion of almost geodesic space is given in Definition 3.3). Since this plays a central role in our construction of metric, we recall the relevant definitions and results below.

We say that a map (not necessarily continuous) $f : (X_1,d_1) \to (X_2,d_2)$ between two metric spaces is a *quasi-isometry* if there exists constants $A, B > 0$ such that

$$A^{-1} d_1(x,y) - A \leq d_2(f(x), f(y)) \leq A d_1(x,y) + A,$$

for all $x, y \in X_1$, and

$$\sup_{x_2 \in X_2} d(x_2, f(X_1)) = \sup_{x_2 \in X_2} \inf_{x_1 \in X_1} d(x_2, f(x_1)) \leq B.$$

Definition 3.1 A *distortion function* is a homeomorphism of $[0,\infty)$ onto itself. Let η be a distortion function. A map $f : (X_1,d_1) \to (X_2,d_2)$ between metric spaces is said to be *η-quasisymmetric*, if f is a homeomorphism and

$$\frac{d_2(f(x), f(a))}{d_2(f(x), f(b))} \leq \eta\left(\frac{d_1(x,a)}{d_1(x,b)}\right) \tag{3.2}$$

for all triples of points $x, a, b \in X_1, x \neq b$. We say f is a *quasisymmetry* if it is η-quasisymmetric for some distortion function η. We say that metric spaces

(X_1, d_1) and (X_2, d_2) are quasisymmetric, if there exists a quasisymmetry $f : (X_1, d_1) \to (X_2, d_2)$. We say that metrics d_1 and d_2 on X are *quasisymmetric* (or, d_1 is quasisymmetric to d_2), if the identity map $\mathrm{Id} : (X, d_1) \to (X, d_2)$ is a quasisymmetry. A quasisymmetry $f : (X_1, d_1) \to (X_2, d_2)$ is said to be a *power quasisymmetry*, if there exists $\alpha > 0, \lambda \geq 1$ such that f is $\eta_{\alpha,\lambda}$-quasisymmetric, where

$$\eta_{\alpha,\lambda}(t) = \begin{cases} \lambda t^{1/\alpha}, & \text{if } 0 \leq t < 1, \\ \lambda t^{\alpha}, & \text{if } t \geq 1. \end{cases}$$

Recall from Definition 1.1 that the *conformal gauge* of a metric space (X, d) is defined as

$$\mathcal{J}(X, d) := \{\theta \colon X \times X \to [0, \infty) \mid \theta \text{ is a metric on } X, \\ d \text{ is quasisymmetric to } \theta\}.$$

Bi-Lipschitz maps are the simplest examples of quasi-symmetric maps. Recall that a map $f : (X_1, d_1) \to (X_2, d_2)$ is said to be *bi-Lipschitz*, if there exists $C \geq 1$,

$$C^{-1} d_1(x, y) \leq d_2(f(x), f(y)) \leq C d_1(x, y), \quad \text{for all } x, y \in X_1.$$

Two metrics $d_1, d_2 : X \times X \to [0, \infty)$ on X are said to be *bi-Lipschitz equivalent* if the identity map $\mathrm{Id} : (X, d_1) \to (X, d_2)$ is bi-Lipschitz.

We collect a few useful facts about quasisymmetric maps.

Proposition 3.2 ([75, Lemma 1.2.18] and [40, Proposition 10,8]) *Let the identity map* $\mathrm{Id} : (X, d_1) \to (X, d_2)$ *be an* η*-quasisymmetry for some distortion function* η*. By* $B_i(x, r)$ *we denote the open ball in* (X, d_i) *with center* x *and radius* $r > 0$*, for* $i = 1, 2$*.*

(a) For all $A \geq 1, x \in X, r > 0$*, there exists* $s > 0$ *such that*

$$B_2(x, s) \subset B_1(x, r) \subset B_1(x, Ar) \subset B_2(x, \eta(A)s). \tag{3.3}$$

In (3.3)*, s can be defined as*

$$s = \sup\{0 \leq s_2 < 2\,\mathrm{diam}(X, d_2) : B_2(x, s_2) \subset B_1(x, r)\} \tag{3.4}$$

Conversely, for all $A > 1, x \in X, r > 0$*, there exists* $t > 0$ *such that*

$$B_1(x, r) \subset B_2(x, t) \subset B_2(x, At) \subset B_1(x, A_1 r), \tag{3.5}$$

where $A_1 = \tilde{\eta}(A)$ *and* $\tilde{\eta}(t)$ *is the distortion function given by* $\tilde{\eta}(t) = 1/\eta^{-1}(t^{-1})$. *In* (3.5)*,* t *can be defined as*

$$At = \sup\{0 \le r_2 < 2\operatorname{diam}(X, d_2) : B_2(x, Ar_2) \subset B_1(x, A_1 r)\}. \tag{3.6}$$

(b) If $A \subset B \subset X$ *such that* $0 < \operatorname{diam}(A, d_1) \le \operatorname{diam}(B, d_1) < \infty$*, then* $0 < \operatorname{diam}(A, d_2) \le \operatorname{diam}(B, d_2) < \infty$ *and*

$$\frac{1}{2\eta\left(\frac{\operatorname{diam}(B,d_1)}{\operatorname{diam}(A,d_1)}\right)} \le \frac{\operatorname{diam}(A, d_2)}{\operatorname{diam}(B, d_2)} \le \eta\left(\frac{2\operatorname{diam}(A, d_1)}{\operatorname{diam}(B, d_1)}\right). \tag{3.7}$$

Definition 3.3 A metric space (X, d) is *k-almost geodesic*, if for every $x, y \in X$ and every $t \in [0, d(x, y)]$, there is some $z \in X$ with $|d(x, z) - t| \le k$ and $|d(y, z) - (d(x, y) - t)| \le k$. We say that a metric space is *almost geodesic* if it is k-almost geodesic for some $k \ge 0$. We recall that quasi-isometry between almost geodesic hyperbolic spaces induces a quasisymmetry between their boundaries.

Proposition 3.4 ([19, Theorem 6.5 and Proposition 6.3]) *Let* (Z_1, d_1) *and* (Z_2, d_2) *be two almost geodesic,* δ*-hyperbolic metric spaces. Let* $f : (Z_1, d_1) \to (Z_2, d_2)$ *denote quasi-isometry.*

(a) If $\{x_i\} \subset Z_1$ *converges at infinity, then* $\{f(x_i)\} \subset Y$ *converges at infinity. If* $\{x_i\}$ *and* $\{y_i\}$ *are equivalent sequences in* X *converging at infinity, then* $\{f(x_i)\}$ *and* $\{f(y_i)\}$ *are also equivalent.*
(b) If $a \in \partial Z_1$ *and* $\{x_i\} \in a$*, let* $b \in \partial Z_2$ *be the equivalence class of* $\{f(x_i)\}$*. Then* $\partial f : \partial Z_1 \to \partial Z_2$ *is well-defined, and is a bijection.*
(c) Let $p_1 \in Z_1$ *be a base point in* Z_1*, and let* $f(p_1)$ *be a corresponding base point in* Z_2*. Let* ρ_1, ρ_2 *denote visual metrics (with not necessarily the same visual parameter) on* $\partial Z_1, \partial Z_2$ *with base points* $p_1, f(p_1)$ *respectively. Then the induced boundary map* $\partial f : (\partial Z_1, \rho_1) \to (\partial Z_2, \rho_2)$ *is a power quasisymmetry.*

Remark 3.5 The distortion function η for the quasisymmetry ∂f in (c) above can be chosen to depend only on the constants associated with the quasi-isometry $f : Z_1 \to Z_2$ and the constants associated with the properties of being almost geodesic and Gromov hyperbolic for Z_1, Z_2.

3.2 Hyperbolic filling of a compact metric space

Given a compact metric space (X, d), one can construct an almost geodesic, hyperbolic space whose boundary equipped with a visual metric can be identified with (X, d). We assume further that (X, d) is doubling and uniformly perfect. Recall that a metric space (X, d) is said to be K_D*-doubling* if any ball $B(x, r)$ can be covered by K_D balls of radius $r/2$, and be *doubling* or satisfy the *metric doubling property* if it is K_D-doubling for some $K_D \in \mathbb{N}$. A metric space (X, d) is said to be K_P*-uniformly perfect* if $B(x, r) \setminus B(x, r/K_P) \neq \emptyset$ for any ball $B(x, r)$ such that $X \setminus B(x, r) \neq \emptyset$, and *uniformly perfect* if it is K_P-uniformly perfect for some $K_P \in (1, \infty)$.

We recall the notion of hyperbolic filling due to Bourdon and Pajot [20], based on a similar construction due to Elek [32]. We recall the definition in [23]. Let (X, d) be a compact, doubling, uniformly perfect, metric space. For a ball $B = B(x, r)$ and $\alpha > 0$, by $\alpha \cdot B$ we denote the ball $B(x, \alpha r)$. It is possible that balls with different centers and/or radii denote the same set. For this reason, we adopt the convention that a ball B comes with a predetermined center and radius denoted by x_B and r_B respectively. We fix two parameter $a > 8$ and $\lambda \geq 3$. The parameters a and λ are respectively called the *vertical* and *horizontal* parameters of the hyperbolic filling. For each $n \geq 0$, let $\mathcal{S}_n$ denote a finite covering of X by open balls such that for all $B \in \mathcal{S}_n$, there exists a center $x_B \in X$ such that

$$B = B(x_B, 2a^{-n}), \tag{3.8}$$

and for any distinct pair $B \neq B'$ in $\mathcal{S}_n$, we have

$$B(x_B, a^{-n}/2) \cap B(x_{B'}, a^{-n}/2) = \emptyset. \tag{3.9}$$

We assume that

$$\mathcal{S}_0 = \{X\} \tag{3.10}$$

is a singleton (by scaling the metric if necessary). We remark that the assumption (3.10) is just for convenience. For arbitrary (but finite) diameter, we choose $n_0 \in \mathbb{Z}$ such that $a^{-n_0} > \mathrm{diam}(X, d) \geq a^{-n_0-1}$. For the general compact case we replace 0 with n_0, so that we have coverings $\mathcal{S}_k$ for all $k \geq n_0$ such that $\mathcal{S}_k$ is a covering by 'almost' pairwise disjoint balls of radii roughly a^{-k} as given in (3.8) and (3.9).

We construct a graph whose vertex set is $\mathcal{S} = \coprod_{n=0}^{\infty} \mathcal{S}_n$. Next, we construct a tree structure of *vertical edges* on $\mathcal{S}$. For each $n \geq 0$, we partition $\mathcal{S}_{n+1}$ into pairwise disjoint sets $\{T_n(B) : B \in \mathcal{S}_n\}$ indexed by $\mathcal{S}_n$, with $\mathcal{S}_{n+1} = \coprod_{B \in \mathcal{S}_n} T_n(B)$ satisfying the following property:

$$\text{if } B' \in T_n(B) \text{ for some } B \in \mathcal{S}_n,\ B' \in \mathcal{S}_{n+1},\ \text{then } d(x_{B'}, A_n) = d(x_{B'}, x_B), \tag{3.11}$$

where $A_n = \{x_B : B \in \mathcal{S}_n\}$ denotes the set of centers of the balls in $\mathcal{S}_n$. In other words, if $B' \in T_n(B)$, then $x_B \in A_n$ is a minimizer to the distance between $x_{B'}$ and A_n. Since such a minimizer always exists, there exists a (not necessarily unique) partition $\{T_n(B) : B \in \mathcal{S}_n\}$ of $\mathcal{S}_{n+1}$ for all $n \geq 0$. We call the elements of $T_n(B)$ as the *children* of B. From now on, let us fix one such partition $\{T_n(B) : B \in \mathcal{S}_n\}$ for each $n \geq 0$. We say that there exist a *vertical edge* between two sets $B, B' \in \mathcal{S}$, if there exists $n \geq 0$ such that either $B \in \mathcal{S}_n$, $B' \in \mathcal{S}_{n+1} \cap T_n(B)$ or $B' \in \mathcal{S}_{n+1}$, $B \in \mathcal{S}_{n+1} \cap T_n(B')$; in other words, one of them is a child of the other. Note that the vertical edges form a tree on the vertex set $\mathcal{S}$, with base point (or root) w, where $\mathcal{S}_0 = \{w\}$. The unique path from the base point to a vertex $B \in \mathcal{S}$ denotes the genealogy $g(B)$. More precisely, we define the genealogy $g(B)$ as (B) as

$$g(B) = \begin{cases} (B), & \text{if } B \in \mathcal{S}_0, \\ (B_0, B_1, \ldots, B_n), & \text{if } B = B_n \in \mathcal{S}_n, n \geq 1, \text{ and} \\ & \quad B_{i+1} \in T(B_i), \text{ for } i = 0, \ldots, n-1. \end{cases}$$

In the above definition, if $0 \leq i < n$, we denote the vertex $B_i \in \mathcal{S}_i$ by $g(B)_i$. If $B \in \mathcal{S}_n$, and $l > n$, we define $\mathcal{D}_l(B)$ as the descendants of B in the generation l

$$\mathcal{D}_l(B) := \left\{B' \in \mathcal{S}_l : g(B')_n = B\right\}. \tag{3.12}$$

For $B \in \mathcal{S}_n$, we denote $\cup_{l \geq n+1} \mathcal{D}_l(B)$ by $\mathcal{D}(B)$ which are the descendants of B.

Using the horizontal parameter $\lambda \geq 3$, we define another family of edges on the vertex set $\mathcal{S}$ call the *horizontal edges*. We say $B \sim B'$ if there exists $n \geq 0$ such that $B, B' \in \mathcal{S}_n$ and $\lambda \cdot B \cap \lambda \cdot B' \neq \emptyset$. We say that there is a *horizontal edge* between $B, B' \in \mathcal{S}$, if $B \sim B'$ and they are distinct (so as to avoid self-loops).

Definition 3.6 (Hyperbolic filling) Let $\mathcal{S}_d = (\mathcal{S}, E)$ denote the graph with vertices in $\mathcal{S}$ and whose edges E are obtained by the taking the union of horizontal and vertical edges. With a slight abuse of notation, we often view $\mathcal{S}_d$ as a metric space equipped with the (combinatorial) graph distance, which we denote by $D_{\mathcal{S}} : \mathcal{S} \times \mathcal{S} \to \mathbb{Z}_{\geq 0}$. The metric space $\mathcal{S}_d = (\mathcal{S}, D_{\mathcal{S}})$ is almost geodesic and hyperbolic [20, Proposition 2.1]. The metric space $\mathcal{S}_d$ is said to be a *hyperbolic filling* of (X, d).

We refer to Sect. 3.3 for a construction of hyperbolic filling. Note that the hyperbolic filling is not unique as we make an arbitrary choice of covering. Even if the covering is fixed, the choice of children $T_n(B)$ is not necessarily unique. Nevertheless, any two hyperbolic fillings (with possibly different parameters) of a metric space are quasi-isometric to each other [20, Corollaire 2.4].

We fix the base point of $\mathcal{S}_d$ to be $w \in \mathcal{S}$, where $\{w\} = \mathcal{S}_0$. We now define a map $p : X \to \partial\mathcal{S}_d$ that identifies X with the boundary of $\mathcal{S}_d$ as follows. For each $x \in X$, choose a sequence $\{B_i\}$ with $x \in B_i \in \mathcal{S}_i$, $i \in \mathbb{N}$. Then it is easy to see that the sequence $\{B_i\}$ converges at infinity. Let $p(x) \in \partial\mathcal{S}_d$ denote the equivalence class containing $\{B_i\}$.

The map p is a bijection and its inverse $p^{-1} : \partial\mathcal{S}_d \to X$ can be described as follows. For any $a \in \partial\mathcal{S}_d$, and for any $\{B_i\} \in a$, the corresponding sequence of centers $\left\{x_{B_i}\right\}$ is a convergent sequence in X, and the limit is $p^{-1}(a) = \lim_{i\to\infty} x_{B_i}$. The map p^{-1} is well-defined; that is, if $\{B_i\}$ and $\left\{B_i'\right\}$ are equivalent sequences that converge at infinity, then $\lim_{i\to\infty} x_{B_i} = \lim_{i\to\infty} x_{B_i'}$.

We summarize the properties of the hyperbolic filling $\mathcal{S}_d$ and its boundary $\partial\mathcal{S}_d$ as follows:

Proposition 3.7 ([20, Proposition 2.1]) *Let (X, d) denote a compact, doubling, uniformly perfect metric space. Let $\mathcal{S}_d$ denote a hyperbolic filling with vertical parameter $a > 1$, and horizontal parameter $\lambda \geq 3$. Then $\mathcal{S}_d$ is almost geodesic Gromov hyperbolic space. The map $p : X \to \partial\mathcal{S}_d$ is a homeomorphism between X and $\partial\mathcal{S}_d$. If we choose the base point $w \in \mathcal{S}_d$ as the unique vertex in $\mathcal{S}_0$, then there exists $K > 1$ such that*

$$K^{-1} a^{-(p(x)|p(y))_w} \leq d(x, y) \leq K a^{-(p(x)|p(y))_w}$$

for all $x, y \in X$.

By the above proposition we can recover the metric space (X, d) from its hyperbolic filling $\mathcal{S}_d$ with horizontal parameter λ and vertical parameter a (up to bi-Lipschitz equivalence) as the boundary $\partial\mathcal{S}_d$ equipped with a visual metric with base point w and visual parameter a. There is a minor gap in [20] as pointed out in [18, Section 4] and [15]. We remark that the horizontal parameter λ was chosen to be 1 in [20]. If $\lambda = 1$, then the hyperbolic filling need not be Gromov hyperbolic [15, Example 8.8]. As pointed out in [18], if $\lambda > 1$ such problems do not arise.

For technical reasons following [23, (2.8)], we will often assume that

$$\lambda \geq 32, \quad a \geq 24(\lambda \vee K_P), \tag{3.13}$$

where K_P is such that (X, d) is K_P-uniformly perfect.

3.3 Construction of hyperbolic fillings

Since the metric spaces we deal with need not be compact, we need a suitable substitute for hyperbolic fillings. To circumvent this difficulty, we view the metric space as an increasing union of compact spaces and construct a sequence of hyperbolic fillings. Quasisymmetric maps and doubling measures have nice compactness properties that persist under such limits.

We recall the notion of net in a metric space.

Definition 3.8 Let (X, d) be a metric space and let $\varepsilon > 0$. A subset N of X is called an *ε-net* in (X, d) if the following two conditions are satisfied:

1. (Separation) N is *ε-separated* in (X, d), i.e., $d(x, y) \geq \varepsilon$ for any $x, y \in N$ with $x \neq y$.
2. (Maximality) If $N \subset M \subset X$ and M is ε-separated in (X, d), then $M = N$.

In the lemma below, we recall a standard construction of hyperbolic filling and some of its properties.

Lemma 3.9 (Cf. [23, Lemma 2.2] and [50, Theorem 2.1]) *Let (X, d) be a complete, K_P-uniformly perfect, K_D-doubling metric space such that either* $\mathrm{diam}(X, d) = \frac{1}{2}$ *or ∞. Let $a > 8$ and let $x_0 \in X$. Let N_0 be a 1-net in (X, d) such that $x_0 \in N_0$. Define inductively the sets N_k for $k \in \mathbb{N}$ such that*

$$N_{k-1} \subset N_k, \text{ and } N_k \text{ is } a^{-k}\text{-net in } (X, d), \text{ for all } k \in \mathbb{N}, .$$

For $k < 0$ and $k \in \mathbb{Z}$, we define N_k to be a a^{-k}-net in (N_{k+1}, d) such that $x_0 \in N_k$ for all $k \in \mathbb{Z}$ (Note that N_k need not be a^{-k}-net in (X, d) for $k < 0$). For each $x \in N_k$ and $k \in \mathbb{Z}$, we pick a predecessor *$y \in N_{k-1}$ such that y is a closest point to x in N_{k-1} (by making a choice if there is more than one closest point); that is $y \in N_{k-1}$ satisfies*

$$d(x, y) = \min_{z \in N_{k-1}} d(x, z).$$

For any $x \in N_k$, $k \in \mathbb{Z}$, we denote its predecessor as defined above by $P(x) \in N_{k-1}$.

(a) For all $k \in \mathbb{Z}$, and for any two distinct points $x, y \in N_k$, we have

$$B(x, a^{-k}/2) \cap B(y, a^{-k}/2) = \emptyset. \tag{3.14}$$

We have the following covering property:

$$\cup_{x \in N_k} B(x, a^{-k}) = X, \quad \text{for all } k \geq 0, \tag{3.15}$$

$$\cup_{x \in N_k} B(x, (1 - a^{-1})^{-1} a^{-k}) = X, \quad \text{for all } k \in \mathbb{Z}. \tag{3.16}$$

In particular, if $\operatorname{diam}(X, d) = \frac{1}{2}$*, the coverings* $\mathcal{S}_n = \left\{B(y, (1-a^{-1})^{-1} a^{-n}) \mid y \in N_n\right\}$ *for all* $n \geq 0$ *is a covering that satisfies* (3.8) *and* (3.9)*. For any* $n \geq 0$ *and for any* $B = B(x_B, a^{-n}) \in \mathcal{S}_n$*, the sets*

$$T_n(B) = \left\{B(y, a^{-n-1}) \mid y \in N_{k+1} \text{such that } x_B = P(y)\right\}$$

forms a partition of $\mathcal{S}_{n+1}$ *as required by* (3.11)*.*

(b) Let a, λ *satisfy* (3.13)*. Let* $y \leq N_{k+1}$ *be such that*

$$B(y, (1-a^{-1})^{-1} a^{-k-1}) \cap B(P(y), a^{-k}/3) \neq \emptyset.$$

Then for any $z \in N_{k+1}$ *such that* $d(y, z) < 2(1-a^{-1})^{-1} a^{-k-1}$*, we have* $P(y) = P(z)$*. (In other words, y corresponds to the center of a non-peripheral ball in* $\mathcal{S}_{k+1}$ *as given in Definition* 3.23*).*

(c) Let $k \in \mathbb{Z}$ *and* $y \in N_k$*. Let* $D_k(y)$ *denote the set of descendants of y defined by*

$$D_k(y) = \{y\} \cup \left\{z \in N_l \mid \text{such that } l > k \text{ and } P^{l-k}(z) = y\right\}. \tag{3.17}$$

Then

$$\overline{B}(y, (1-a^{-1})^{-1} a^{-k}) \supset \overline{D_k(y)} \supset B(y, (2^{-1} - (a-1)^{-1}) a^{-k}). \tag{3.18}$$

The space $\overline{D_k(y)}$ *with the restricted metric d is* K_D^2*-doubling and* K_P'*-uniformly perfect, where* $K_P' = 2aK_P(1-a^{-1})^{-1}(2^{-1} - (a-1)^{-1})^{-1}$*.*

Proof (a) The properties (3.14) and (3.15) follow from the separation and maximality properties of the a^{-k}-net N_k in X respectively. We use the notation $P^k(y)$ denote the k-predecessor of y (for example, $P^2(y) = P(P(y))$). To show (3.16), by (3.15) it suffices to consider the case $k < 0$. By (3.15), for any $y \in X$ there exists $y_0 \in N_0$ such that $d(y_0, y) < 1$. Define $y_l = P^{-l}(y)$ for all $l < 0$. Since $d(y_l, y_{l+1}) < a^{-l}$ for all $l < 0$, we have

$$\begin{aligned} d(y_k, y) &\leq d(y_0, y) + \sum_{l=-1}^{k} d(y_l, y_{l+1}) \\ &< \sum_{l=0}^{k} a^{-l} = (1-a^{-1})^{-1}(a^{-k} - a^{-1}) < (1-a^{-1})^{-1} a^{-k}. \end{aligned} \tag{3.19}$$

Since $y \in X$ is arbitrary and $y_k \in N_k$, we have (3.16).

(b) By the triangle inequality, we have

$$\begin{aligned} d(z, P(y)) &\le d(z, y) + d(y, P(y)) \\ &\le (2\lambda + 1)(1 - a^{-1})^{-1} a^{-k-1} + \frac{1}{3} a^{-k} \\ &< a^{-k}/2 \quad \text{(by (3.13))} \end{aligned}$$

By (3.15) and $d(z, P(y)) < a^{-k}/2$, we conclude that $P(z) = P(y)$.

(c) By (3.15) and triangle inequality, we have

$$d(y, z) \ge a^{-k}/2, \quad \text{for all } z \in N_{k+1} \setminus D(y) \text{ and for all } y \in N_k. \tag{3.20}$$

By (3.19), we have

$$\overline{D(z)} \subset \overline{B}(z, (1 - a^{-1})^{-1} a^{-k-1}), \quad \text{for all } z \in N_{k+1}. \tag{3.21}$$

Since $\bigcup_{w \in N_l} \overline{D(w)}$ is dense and closed (by the doubling property), we have

$$\bigcup_{w \in N_l} \overline{D(w)} = X, \quad \text{for all } l \in \mathbb{Z}. \tag{3.22}$$

Combining (3.20), (3.21), (3.22) and using triangle inequality, we obtain (3.18).

Next, we show that $\overline{D(y)}$ is K_D-doubling. More generally, we show that any subset $Y \subset X$ is K_D^2-doubling. Let $B(x, r) \cap Y, x \in Y$ be an arbitrary ball in Y. Since (X, d) is K_D-doubling, the ball $B(x, r)$ can be covered by N balls $B(x_i, r/4), i = 1, \dots, N$, where $N \ge K_D^2$. If $B(x_i, r/4) \cap Y \ne \emptyset$, we choose $y_i \in B(x_i, r/4) \cap Y$, so that $B(x_i, r/4) \subset B(y_i, r/2)$. Hence all such balls $B(y_i, r/2) \cap Y$ cover $B(x, r) \cap Y$.

Let $B(x, r) \cap \overline{D(y)}$ be an arbitrary ball in $\overline{D(y)}$ such that $x \in \overline{D(y)}$ and $B(x, r) \cap \overline{D(y)} \ne \overline{D(y)}$. Let $n \in \mathbb{Z}$ be the unique integer such that

$$(1 - a^{-1})^{-1} a^{-n} < r \le (1 - a^{-1})^{-1} a^{-n+1}.$$

Since $\overline{D(y)} = \cup_{z \in N_n \cap D(y)} \overline{D(z)}$ for all $n \ge k$, by (3.18), there exists $z \in D(y) \cap N_n$ such that

$$d(z, x) \le (1 - a^{-1})^{-1} a^{-n} < r. \tag{3.23}$$

Since (X,d) is K_P-uniformly perfect, and using (3.18) and $B(z,(2^{-1}-(a-1)^{-1})a^{-n}) \neq X$, there exists $w \in \overline{D(y)}$ such that

$$(2^{-1}-(a-1)^{-1})^{-1}a^{-n} > d(w,z) \geq \frac{1}{K_P}(2^{-1}-(a-1)^{-1})a^{-n}. \tag{3.24}$$

We consider two cases, depending on whether or not $d(z,x) < \frac{1}{2}a^{-n}$. If $d(z,y) \geq \frac{1}{2}a^{-n}$, then

$$r > d(z,x) \geq \frac{1}{2}a^{-n} \geq \frac{r}{2a((1-a^{-1})^{-1}}. \tag{3.25}$$

On the other hand, if $d(z,x) < \frac{1}{2}a^{-n}$, then

$$\begin{aligned} d(w,x) \vee d(z,x) &\leq d(z,w) + d(z,x) \\ &< (2^{-1}-(a-1)^{-1})^{-1}a^{-n} + \frac{1}{2}a^{-n} < a^{-n} < r. \end{aligned}$$

Hence by (3.24), if $d(z,x) < \frac{1}{2}a^{-n}$, we have

$$\begin{aligned} d(w,x) \vee d(z,x) &\geq \frac{1}{2}d(w,z) \geq \frac{1}{2K_P}(2^{-1}-(a-1)^{-1})a^{-n} \\ &\geq \frac{r}{2aK_P(1-a^{-1})^{-1}(2^{-1}-(a-1)^{-1})^{-1}}. \end{aligned} \tag{3.26}$$

By (3.25) and (3.26), $\overline{D(y)}$ is $2aK_P(1-a^{-1})^{-1}(2^{-1}-(a-1)^{-1})^{-1}$-uniformly perfect.

□

Definition 3.10 (Extended hyperbolic filling) Let (X,d) be a complete, K_P-uniformly perfect doubling metric space such that either $\operatorname{diam}(X,d) = \frac{1}{2}$ or ∞. Let $a > 8, \lambda \geq 32$ be constants that satisfy (3.13). Let $x_0 \in X$ and consider the sets $N_k, k \in \mathbb{Z}$ as defined in Lemma 3.9. Define

$$\mathcal{S}_k = \left\{ B(x, 2a^{-k}) : x \in N_k \right\}, \quad k \in \mathbb{Z}.$$

For any $k \in \mathbb{Z}$ and for any pair of distinct balls $B, B' \in \mathcal{S}_k$, we say that there is a *horizontal edge* between B and B' (denoted as $B \sim B'$) if and only if $\lambda \cdot B \cap \lambda \cdot B' \neq \emptyset$. For any $k \in \mathbb{Z}$ and for any $B(x,2a^{-k}) \in \mathcal{S}_k$, $B(y,2a^{-k-1}) \in \mathcal{S}_{k+1}$, we say that there is a *vertical edge* between $B(x,2a^{-k})$ and $B(y,2a^{-k-1})$, if x is the predecessor of y (as defined in Lemma 3.9). We define a graph

(V, E) with vertex set $V = \coprod_{k \in \mathbb{Z}} \mathcal{S}_k$ and the edge set E defined by the union of horizontal and vertical edges. This graph is called the *extended hyperbolic filling* of (X, d) with horizonal parameter λ and vertical parameter a.

If (X, d) is compact, the subgraph of the extended hyperbolic filling induced by $\mathcal{S} = \coprod_{k \in \mathbb{Z}_{\geq 0}} \mathcal{S}_k$ forms a hyperbolic filling as given in Definition 3.6.

On the other hand, if (X, d) is non-compact, we view X as an increasing limit of compact spaces $\overline{D_l(x_0)}$ as $l \to -\infty$, where $D_k(x_0)$ is as defined in (3.17). For any $k, l \in \mathbb{Z}, l \leq 0, k \geq l$, we define

$$\mathcal{S}_k^l = \left\{ B(x, 2a^{-k}) \cap \overline{D_l(x_0)} : x \in N_k \cap D_l(x_0) \right\}.$$

We define a graph with vertex set $\mathcal{S}^l = \coprod_{k \geq l, k \in \mathbb{Z}} \mathcal{S}_k^l$, whose edges are the union of horizonal and vertical edges. In this case, the vertical edges are defined using predecessor relation as above and the horizontal edges are defined with respect to the space $\overline{D_l(x_0)}$. That is $B \cap \overline{D_l(x_0)}$, $B' \cap \overline{D_l(x_0)} \in \mathcal{S}_k^l$ share a horizontal edge if and only if $\lambda \cdot B \cap \lambda \cdot B' \cap \overline{D_l(x_0)} \neq \emptyset$. This graph with vertex set $\mathcal{S}^l$ can be viewed as a hyperbolic filling of the compact space $\overline{D_l(x_0)}$. In the non-compact case, we think of the sequence of hyperbolic fillings defined with vertex set $\mathcal{S}^l$ as 'converging' to the extended hyperbolic filling defined above as $l \to -\infty$.

3.4 Combinatorial description of the conformal gauge

The purpose of this section is to recall a combinatorial description of the conformal gauge essentially due to M. Carrasco Piaggo [23]. In this section, we fix a compact, doubling, uniformly perfect metric space (X, d) and a hyperbolic filling $\mathcal{S}_d = (\mathcal{S}, D_{\mathcal{S}})$ with horizontal parameter $\lambda \geq 8$ and vertical parameter $a > 1$ that satisfies (3.10).

Propositions 3.4 and 3.7 suggest the following strategy to construct metrics that are in the conformal gauge of (X, d). By changing the metric of the hyperbolic filling $\mathcal{S}_d$ to another metric that is almost geodesic and bi-Lipschitz (in particular, quasi-isometric), every visual metric of its boundary is changed to a metric in the conformal gauge of (X, d). Perhaps surprisingly, all metrics in the conformal gauge can be obtained in this manner (up to a bi-Lipschitz map) as explained in Theorem 3.14.

The change of metric in a hyperbolic filling is done using a *weight function* $\rho : \mathcal{S} \to (0, 1)$ on its vertex set. We define

$$\pi(B) = \prod_{B' \in g(B)} \rho(B'). \tag{3.27}$$

A *path* $\gamma = \{B_i\}_{i=1}^N$ in $\mathcal{S}_d$ is a sequence of vertices such that there is an edge between B_i and B_{i+1} for all $i = 1, \dots, N-1$. In this case, we say that γ is a path from B_1 to B_N. A path is said to be *simple*, if no two vertices in the path are the same. A path is said to be *horizontal* (resp. *vertical*), if all the edges in the path are horizontal (resp. vertical). We define the ρ-length of a path $\gamma = \{B_i\}_{i=1}^N$ by

$$L_\rho(\gamma) = \sum_{i=1}^N \pi(B_i), \tag{3.28}$$

where π is as defined in (3.27). For points $x, y \in X$ and $n \in \mathbb{N}$, the set of paths $\Gamma_n(x, y)$ is defined as

$$\Gamma_n(x, y) = \left\{ \gamma = \{B_i\}_{i=1}^k \;\middle|\; \begin{array}{l} \gamma \text{ is a path from } B_1 \text{ to } B_k, x \in B_1, \\ y \in B_k, B_1 \in \mathcal{S}_n, B_k \in \mathcal{S}_n \end{array} \right\}. \tag{3.29}$$

We remark that a path $\gamma \in \Gamma_n(x, y)$ need not be a horizontal path. For two distinct points $x, y \in X$ and $\alpha \geq 2$, we define

$$\begin{aligned} m_\alpha(x, y) &= \max\left\{k : B \in \mathcal{S}_k, x \in \alpha \cdot B, y \in \alpha \cdot B\right\}, \\ c_\alpha(x, y) &= \left\{B \in \mathcal{S}_k : k = m_\alpha(x, y), x \in \alpha \cdot B, y \in \alpha \cdot B\right\}, \\ \pi(c_\alpha(x, y)) &= \max_{B \in c_\alpha(x,y)} \pi(B). \end{aligned} \tag{3.30}$$

Assumption 3.11 A weight function $\rho : \mathcal{S} \to (0, 1)$ may satisfy some of the following hypotheses:

(H1) (Quasi-isometry) There exist $0 < \eta_- \leq \eta_+ < 1$ so that $\eta_- \leq \rho(B) \leq \eta_+$ for all $B \in \mathcal{S}$.

(H2) (Gromov product) There exists a constant $K_0 \geq 1$ such that for all $B, B' \in \mathcal{S}$ with $B \sim B' \in \mathcal{S}$, we have

$$\pi(B) \leq K_0 \pi(B'),$$

where π is as defined in (3.27).

(H3) (Visual parameter) There exists $\alpha \in [2, \lambda/4]$ and a constant $K_1 \geq 1$ such that for any pair of points $x, y \in X$, there exists $n_0 \geq 1$ such that if $n \geq n_0$ and γ is a path in $\Gamma_n(x, y)$, then

$$L_\rho(\gamma) \geq K_1^{-1} \pi(c_\alpha(x, y)),$$

where $\Gamma_n(x, y), L_\rho, \pi(c_\alpha(x, y))$ are as defined in (3.29), (3.28), and (3.30) respectively.

The following observation concerns the stability of the above assumption under ‘finite perturbations’.

Remark 3.12 Let $\rho, \rho' : \mathcal{S} \to (0, 1)$ be two different weight functions such that the set $\{B \in \mathcal{S} : \rho(B) \neq \rho'(B)\}$ is finite. Then if ρ satisfies the hypotheses (H1), (H2), and (H3), then so does ρ' (with possibly different constants).

The weight function ρ can be used to define a metric on $\mathcal{S}$ that is bi-Lipschitz equivalent to $D_{\mathcal{S}}$ as we recall below. We summarize the properties of the metric below.

Lemma 3.13 ([23, Lemma 2.3 and Proposition 2.4]) *Let (X, d) be a compact, doubling, uniformly perfect metric space with* $\operatorname{diam}(X, d) = \frac{1}{2}$, *and let $\mathcal{S}_d = (\mathcal{S}, D_{\mathcal{S}})$ denote a hyperbolic filling with parameters λ, a satisfying* (3.13) *and* (3.10). *Let $\rho : \mathcal{S} \to (0, 1)$ be a weight function that satisfies* (H1) *and* (H2). *Then there exists a metric D_ρ on $\mathcal{S}$ such that:*

(a) D_ρ is bi-Lipschitz equivalent to $D_{\mathcal{S}}$; that is there exists $\Lambda \geq 1$ such that

$$\Lambda^{-1} D_{\mathcal{S}}(B, B') \leq D_\rho(B, B') \leq \Lambda D_{\mathcal{S}}(B, B'), \quad \textit{for all } B, B' \in \mathcal{S};$$

(b) any simple vertical path $\gamma = \{B_i\}_{i=1}^n$ joining $B \in \mathcal{S}_m$ and $B' \in \mathcal{S}_{m'}$ satisfies

$$D_\rho(B, B') = \sum_{i=1}^{n-1} D_\rho(B_i, B_{i+1}) = \left| \log \frac{1}{\pi(B)} - \log \frac{1}{\pi(B')} \right|;$$

(c) $(\mathcal{S}, D_\rho)$ is almost geodesic and Gromov hyperbolic.

(d) The identity map $\mathrm{Id} : (\mathcal{S}, D_{\mathcal{S}}) \to (\mathcal{S}, D_\rho)$ *induces the identity map on their boundaries as described in Proposition* 3.4. *That is, a sequence $\{B_i\}$ converges at infinity in $(\mathcal{S}, D_{\mathcal{S}})$ if and only if it converges at infinity in $(\mathcal{S}, D_\rho)$, and any two sequences that converge at infinity in $(\mathcal{S}, D_{\mathcal{S}})$ are equivalent if and only if they are equivalent in $(\mathcal{S}, D_\rho)$. In particular, the bijection $p : X \to \partial(\mathcal{S}, D_{\mathcal{S}})$ described before Proposition* 3.4 *can be viewed as a bijection $\tilde{p} : X \to \partial(\mathcal{S}, D_\rho)$ by composing with the induced identity map above.*

(e) Assume in addition that (H3) *is also satisfied. Let $(\cdot|\cdot)_\rho$ denote the Gromov product on $(\mathcal{S}, D_\rho)$ with base point $w \in \mathcal{S}_0$ extended to its boundary. Define $\tilde{\theta}_\rho : \partial(\mathcal{S}, D_\rho) \times \partial(\mathcal{S}, D_\rho) \to [0, \infty)$ as*

$$\tilde{\theta}_\rho(\tilde{p}(x), \tilde{p}(y)) = \inf \sum_{i=1}^{n-1} e^{-(\tilde{p}(x_i)|\tilde{p}(x_{i+1}))_\rho}, \tag{3.31}$$

where the infimum is over all finite sequence of points $\{x_i\}_{i=1}^n$ *in* X *such that* $n \in \mathbb{N}$, $x_1 = x$, *and* $x_n = y$. *Then* $\tilde{\theta}_\rho$ *is a visual metric on* $\partial(\mathcal{S}, D_\rho)$ *with visual parameter* e. *Moreover, there exists* $K > 1$ *such that*

$$K^{-1} e^{-(\tilde{p}(x)|\tilde{p}(y))_\rho} \le \tilde{\theta}_\rho(\tilde{p}(x), \tilde{p}(y)) \le K e^{-(\tilde{p}(x)|\tilde{p}(y))_\rho},$$
$$K^{-1} \pi(c_\alpha(x, y)) \le \tilde{\theta}_\rho(\tilde{p}(x), \tilde{p}(y)) \le K \pi(c_\alpha(x, y)).$$

Proof (Sketch of the proof) We briefly recall the construction of the metric D_ρ. Let E denote the edge set of the hyperbolic filling and let η_-, η_+, K_0 denote the constants in hypotheses (H1), and (H2). Define a function $\ell_\rho : E \to (0, \infty)$ as

$$\ell_\rho(e) = \begin{cases} 2\max\{-\log(\eta_+), -\log(\eta_-), \log(K_0)\}, & \text{if } e \text{ is a horizontal edge,} \\ \left|\log \frac{\pi(B')}{\pi(B)}\right|, & \text{if } e = (B', B) \text{ is vertical.} \end{cases}$$

Then the distance $D_\rho : \mathcal{S} \times \mathcal{S} \to [0, \infty)$ is defined as

$$D_\rho(B, B') = \inf_\gamma \sum_{i=1}^{N-1} \ell_\rho(e_i),$$

where the infimum is taken over all paths $\gamma = \{B_i\}_{i=1}^N$ where N varies over $\mathbb{N}$, $B_1 = B$, $B_N = B'$ and $e_i = (B_i, B_{i+1})$ is an edge for each $i = 1, \dots, N-1$.

Part (a) is immediate from the definition of D_ρ. Part (b) and (c) are proved in [23, Lemma 2.3]. Part (d) follows from (a), (c) and Proposition 3.4. Part (e) follows from [23, Proposition 2.4]. □

The following theorem provides a combinatorial description of the conformal gauge $\mathcal{J}(X, d)$. In [23, Theorem 1.1], Carrasco Piaggio has provided a combinatorial description of the *Ahlfors regular* conformal gauge

$$\mathcal{J}_{\mathrm{AR}}(X, d) = \{\theta \in \mathcal{J}(X, d) \mid \text{an Ahlfors regular measure } \mu \text{ on } (X, \theta) \text{ exists}\}.$$

In [23, Theorem 1.1] the hypotheses (H1), (H2), (H3) correspond to a combinatorial description of $\mathcal{J}(X, d)$, whereas the hypothesis (H4) corresponds to the existence of an Ahlfors regular measure. This theorem is essentially contained in [23].

Theorem 3.14 (Cf. [23, Theorem 1.1]) *Let* (X, d) *be a compact, doubling, uniformly perfect metric space.*

(a) Let $\mathcal{S}_d = (\mathcal{S}, D_\mathcal{S})$ *denote a hyperbolic filling with parameters* λ, a *satisfying* (3.13) *and* (3.10). *Let* $\rho : \mathcal{S} \to (0, 1)$ *be a weight function*

that satisfies the conditions (H1), (H2), and (H3). Define the metric $\theta_\rho : X \times X \to [0, \infty)$ *as*

$$\theta_\rho(x, y) = \tilde{\theta}_\rho(\tilde{p}(x), \tilde{p}(y)) \quad \text{for } x, y \in X, \tag{3.32}$$

where $\tilde{\theta}_\rho$ *is as defined in (3.31). Then* θ_ρ *satisfies the following properties:*

(i) $\theta_\rho \in \mathcal{J}(X, d)$*; that is* θ_ρ *is quasisymmetric to* d*.*

(ii) there exists $C > 0$ *such that*

$$C^{-1}\pi(c_\alpha(x, y)) \le \theta_\rho(x, y) \le C\pi(c_\alpha(x, y)), \tag{3.33}$$

where α *is the constant in (H3). Furthermore, there exists* $K > 1$ *such that*

$$K^{-1}\pi(B) \le \operatorname{diam}(B, \theta_\rho) \le K\pi(B) \qquad \text{for all } B \in \mathcal{S}. \tag{3.34}$$

(iii) θ_ρ *is a visual metric of the hyperbolic space* $(\mathcal{S}, D_\rho)$ *constructed in Lemma 3.13 in the following sense: there exists* $C > 0$ *such that*

$$C^{-1}\theta_\rho(x, y) \le e^{-(\tilde{p}(x)|\tilde{p}(y))_\rho} \le C\theta_\rho(x, y),$$

where $\tilde{p} : X \to \partial(S, D_\rho)$ *is the bijection described in Lemma 3.13-(d), and* $(\cdot|\cdot)_\rho$ *denotes the Gromov product (extended to the boundary) on the hyperbolic space* $(\mathcal{S}, D_\rho)$ *with base point* $w \in \mathcal{S}_0$*.*

(iv) The distortion function η *of the power quasisymmetry* $\mathrm{Id} : (X, d) \to (X, \theta)$ *can be chosen to depend only on the constants in (H1), (H2), and (H3).*

(b) Conversely, let $\theta \in \mathcal{J}(X, d)$ *be any metric in the conformal gauge. Then there exists a hyperbolic filling* $\mathcal{S}_d = (\mathcal{S}, D_{\mathcal{S}})$ *of* (X, d) *with horizontal parameter* λ*, vertical parameter* a*, and a weight function* $\rho : \mathcal{S} \to (0, 1)$ *that satisfies the hypotheses (H1), (H2), (H3), and such that the metric* θ_ρ *defined in (3.32) is bi-Lipschitz equivalent to* θ*.*

Proof We begin with the proof of (b).

(b) Let $\mathrm{Id} : (X, d) \to (X, \theta)$ be an η-quasisymmetry for some distortion function η.

The definition of the weight function ρ in [23] uses an Ahlfors regular measure. Since there is no such measure available in our setting, the following definition is more suited for our purposes. We normalize the metric θ, so that $\operatorname{diam}(X, \theta) = \frac{1}{2}$. We will define the weight function $\rho : \mathcal{S} \to (0, 1)$ so that

$$\pi(B) = \operatorname{diam}_\theta(B),$$

for all $B \in \mathcal{S}$, where $\mathcal{S}$ is a hyperbolic filling of (X, d) with parameters λ, a. Fix any $\lambda \geq 32$. The vertical parameter $a > 1$ will be determined later in the proof. Hence we define $\rho : \mathcal{S} \to (0, 1)$ as

$$\rho(B) = \begin{cases} \frac{1}{2} & \text{if } B \in \mathcal{S}_0, \\ \frac{\operatorname{diam}_\theta(B)}{\operatorname{diam}_\theta(g(B)_{n-1})} & \text{if } B \in \mathcal{S}_n, n \geq 1. \end{cases}$$

First, we show (H2). Let $B \sim B'$ with $B, B' \in \mathcal{S}_n$. Then choose $y \in \lambda \cdot B \cap \lambda \cdot B'$. By triangle inequality,

$$B \subset B_d(x_B, 2a^{-n}) \subset B_d(y, (\lambda + 2)a^{-n}), \quad B' \subset B_d(y, (\lambda + 2)a^{-n}).$$

By uniform perfectness, and triangle inequality, for any $r < \frac{1}{2}$, $r/K_P \leq \operatorname{diam}_d(B_d(x, r)) \leq 2r$. Therefore by (3.7), we obtain

$$\frac{1}{2\eta(4(\lambda + 2)K_P)} \leq \frac{\operatorname{diam}_\theta(B)}{\operatorname{diam}_\theta(B_d(y, (\lambda + 2)a^{-n}))} \leq \eta(8K_P/(\lambda + 2)).$$

Since the same inequality holds with B replaced with B', we have (H2) with constant

$$K_0 = 2\eta(4(\lambda + 2)K_P)\eta(8K_P/(\lambda + 2)),$$

that depends only on the distortion function η, the constant K_P of uniform perfectness, and the horizontal parameter λ (in particular, does not depend on the vertical parameter a).

Next, we show (H1), which again relies on (3.7). We will choose $a > 2(\lambda + 1)$ large enough so that $\eta_+ = \frac{1}{2}$ in (H1). Clearly this choice works when $B \in \mathcal{S}_0$. If $B = \mathcal{S}_n, n \geq 1$, and by denoting $B' = g(B)_{n-1}$, we have $x_B \in B'$. For $n \geq 2$, we write (the case $n = 1$ is easier and left to the reader)

$$\rho(B) = \frac{\operatorname{diam}_\theta(B)}{\operatorname{diam}_\theta(B')} = \frac{\operatorname{diam}_\theta(B)}{\operatorname{diam}_\theta((4a) \cdot B)} \frac{\operatorname{diam}_\theta((4a) \cdot B)}{\operatorname{diam}_\theta(B')}.$$

Each of the terms can be estimated (from above and below) using (3.7), since by the triangle inequality and $d(x_B, x_{B'}) < 2a^{-n+1}$ we have $B \subset (4a) \cdot B$, and $(4a) \cdot B \supset B(x_B, 2a^{-n-1}) \supset B'$. Hence, we obtain

$$\begin{aligned} \rho(B) &\leq 2\eta\left(\frac{2\operatorname{diam}_d(B)}{\operatorname{diam}_d((4a) \cdot B)}\right)\eta\left(\frac{\operatorname{diam}_d((4a) \cdot B)}{\operatorname{diam}_d(B')}\right) \\ &\leq 2\eta\left(K_p/a\right)\eta\left(16K_P\right) \end{aligned}$$

$$\begin{aligned}\rho(B) &\geq \left[2\eta\left(\frac{\operatorname{diam}_d((4a)\cdot B)}{\operatorname{diam}_d(B)}\right)\eta\left(2\frac{\operatorname{diam}_d(B')}{\operatorname{diam}_d((4a)\cdot B)}\right)\right]^{-1} \\ &\geq \left[2\eta\left(8K_P\right)\eta\left(2K_P/a\right)\right]^{-1}.\end{aligned}$$

First we choose a large enough so that $2\eta\left(K_p/a\right)\eta\left(16K_P\right) \leq \frac{1}{2}$ and (3.13) are satisfied. We set $\eta_- = \left[2\eta\left(8K_P\right)\eta\left(2K_P/a\right)\right]^{-1}$. Hence we obtain (H1).

For (H3), we once again use (3.7), to see that $\pi(B) = \operatorname{diam}_\theta(B)$ is comparable to $\operatorname{diam}_\theta(\lambda \cdot B)$ for all $B \in \mathcal{S}$. More precisely, we have

$$\operatorname{diam}_\theta(B) \leq \operatorname{diam}_\theta(\lambda \cdot B) \leq 2\eta(2\lambda K_P)\operatorname{diam}_\theta(B)$$

for all $B \in \mathcal{S}$. For any path $\gamma = \{B_i\}_{i=1}^m \in \Gamma_n(x, y)$, we choose points $x_i \in \lambda \cdot B_i \cap \lambda \cdot B_{i+1}$, $i = 1, \ldots, m-1$, $x_0 = x$, $x_m = y$ so that

$$\begin{aligned}\theta(x, y) &\leq \sum_{i=0}^{m-1}\theta(x_i, x_{i+1}) \leq \sum_{i=1}^{m}\operatorname{diam}_\theta(\lambda \cdot B_i) \\ &\leq \sum_{i=1}^{m}\operatorname{diam}_\theta(B_i) = 2\eta(2\lambda K_P)L_\rho(\gamma).\end{aligned} \tag{3.35}$$

Fixing $\alpha = 2$, and let $C \in c_2(x, y)$ such that $\pi(c_2(x, y)) = \pi(C)$. Let $m = m_2(x, y)$. Let $B \in \mathcal{S}_{m+1}$ be such that $x \in B$. By definition of $m_2(x, y)$, $y \notin 2 \cdot B$. Therefore $d(x, y) \geq d(x_B, y) - d(x_B, x) \geq a^{-m-1}$. By (3.7), and $\pi(c_2(x, y)) \leq \operatorname{diam}_\theta(2 \cdot C)$ we have

$$\begin{aligned}\pi(c_2(x, y)) &\leq 2\eta\left(\frac{\operatorname{diam}_d(2 \cdot C)}{d(x, y)}\right)\theta(x, y) \\ &\leq 2\eta\left(\frac{8a^{-m}}{a^{-m-1}}\right)\theta(x, y) = 2\eta(8a)\theta(x, y).\end{aligned} \tag{3.36}$$

Combining (3.35) and (3.36) yields (H3) with $\alpha = 2$.

(a) This part is essentially contained in [23]. The hypotheses (H1) and (H2) are used to construct a metric D_ρ on $\mathcal{S}$ as given in Lemma 3.13. If θ_ρ were defined using (3.32), it clearly satisfies the symmetry $\theta_\rho(x, y) = \theta_\rho(y, x)$, and triangle inequality. The role of (H3) is to show that $\theta_\rho(x, y)$ is at least $e^{-(\tilde{p}(x)|\tilde{p}(y))_\rho}$ (up to a constant factor) as explained in Lemma 3.13-(e). The fact that θ_ρ is quasisymmetric to d follows from Lemma 3.13, Propositions 3.7 and 3.4-(c). The statement about the dependence of distortion function η on the constants follow from Remark 3.5.

The estimate (3.34) is also implicitly contained in [23] and is a consequence of (3.33). Choose $x, y \in B$ such that $d(x, y) \geq \operatorname{diam}(B, d)/2$. Since Id :

$(X,d) \to (X,\theta_\rho)$ is an η-quasisymmetry, by (3.7), there exists $C_1 > 0$ such that

$$\theta_\rho(x,y) \le \operatorname{diam}(B,\theta_\rho) \le C_1\theta_\rho(x,y).$$

Since $d(x,y) \ge \operatorname{diam}(B)/2$, B is at a bounded distance in $(\mathcal{S}, D_{\mathcal{S}})$ from any set $C \in c_\alpha(x,y)$. Combining these estimates along with (3.33) and (3.7), we obtain

$$\operatorname{diam}(B,\theta_\rho) \asymp \theta_\rho(x,y) \asymp \pi(c_\alpha(x,y)).$$

□

3.5 Construction of measure using the hyperbolic filling

As in Sect. 3.2, we fix a compact, doubling, uniformly perfect metric space (X,d) with $\operatorname{diam}(X,d) = \frac{1}{2}$, and a hyperbolic filling $\mathcal{S}_d = (\mathcal{S}, D_{\mathcal{S}})$ with horizontal parameter λ and vertical parameter a that satisfy (3.13).

Definition 3.15 (gentle function) Let $\mathcal{C}\colon \mathcal{S} \to (0,\infty)$ and $K \ge 1$. We say that $\mathcal{C}$ is K-*gentle* if

$$K^{-1}\mathcal{C}(B') \le \mathcal{C}(B) \le K\mathcal{C}(B),$$

whenever there is an edge between B and B'. We say that $\mathcal{C}\colon \mathcal{S} \to (0,\infty)$ is *gentle* if it is K-gentle for some $K \ge 1$. The notion of K-gentle function extends to any function $f\colon V \to (0,\infty)$ on a graph $G = (V,E)$. In other words, we say that a function $f\colon V \to (0,\infty)$ is K-gentle if $\log f$ is $(\log K)$-Lipschitz with respect to the graph distance metric.

We sometimes need to distinguish between the horizontal and vertical edges (see Theorem 3.24). We say that $\mathcal{C}\colon \mathcal{S} \to (0,\infty)$ is (K_h, K_v)-*gentle* if

$$K_h^{-1}\mathcal{C}(B') \le \mathcal{C}(B) \le K_h\mathcal{C}(B'),$$

whenever B and B' share a horizontal edge, and

$$K_v^{-1}\mathcal{C}(B') \le \mathcal{C}(B) \le K_v\mathcal{C}(B'),$$

whenever B and B' share a vertical edge. Therefore every (K_h, K_v)-gentle function is $(K_h \vee K_v)$-gentle.

Given a hyperbolic filling $\mathcal{S}$, we need to approximate a ball $B(x,r)$ by a ball in the filling $\mathcal{S}$. We introduce this notion in the following definition.

Definition 3.16 Let (X, d) be a doubling metric space. Let $(\mathcal{S}, D_{\mathcal{S}})$ be a hyperbolic filling of (X, d) with parameters a, λ that satisfy (3.13) as constructed in Lemma 3.9-(a). By Lemma 3.9-(a), given a ball $B(x, r) \neq X$, there exists $n \in \mathbb{Z}$ and $B \in \mathcal{S}_n$ such that

$$2a^{-n-1} \leq r < 2a^{-n}, \quad \text{and} \quad d(x_B, x) < 2a^{-n}. \tag{3.37}$$

We define

$$\mathcal{A}_{\mathcal{S}}(B(x, r)) = \left\{ B \in \mathcal{S} : n \in \mathbb{Z}_{\geq 0} \text{ and } B \in \mathcal{S}_n \text{ satisfy (3.37)} \right\}. \tag{3.38}$$

We remark that if $B, B' \in \mathcal{A}_{\mathcal{S}}(B(x, r))$, then $x \in B \cap B' \neq \emptyset$ and hence B and B' share a horizontal edge.

Often, the measures in this work will satisfy the following volume doubling and reverse volume doubling properties.

Definition 3.17 (Volume doubling and Reverse volume doubling properties) Let (X, d) be a metric space and let μ be a Borel measure on X.

(a) We say that μ satisfies the *volume doubling property* VD, or μ is a *doubling measure* on (X, d), or (X, d, μ) is VD, if there exists $C_D \in (1, \infty)$ such that

$$0 < \mu(B(x, 2r)) \leq C_D \mu(B(x, r)) < \infty \quad \text{for all } x \in X, r \in (0, \infty). \tag{VD}$$

(b) We say that μ satisfies the *reverse volume doubling property* RVD, or (X, d, μ) is RVD, if there exist $C_1, C_2 \in (1, \infty)$, $\alpha \in (0, \infty)$ such that

$$\mu(B(x, R)) \geq C_1^{-1} \left(\frac{R}{r} \right)^{\alpha} \mu(B(x, r)) \tag{RVD}$$

for all $x \in X, 0 < r \leq R < \operatorname{diam}_d(X)/C_2$.

Remark 3.18 We recall the following connections between the doubling and uniform perfectness properties of a metric space (X, d) and the volume doubling and reverse volume doubling properties.

(a) If μ satisfies VD on (X, d), then (X, d) is a doubling metric space. Conversely, every complete doubling metric space admits a measure that satisfies VD [40, Theorem 13.3]. The constant 2 in the definition of VD is essentially arbitrary, as VD implies

$$\frac{\mu(B(x, R))}{\mu(B(x, r))} \leq C_D \left(\frac{R}{r} \right)^{\alpha}, \text{ for all } x \in X, 0 < r \leq R, \text{ where } \alpha = \log_2 C_D. \tag{3.39}$$

(b) Let μ be a measure that satisfies VD on (X, d). Then μ satisfies RVD if and only if (X, d) is uniformly perfect [40, Exercise 13.1].

We introduce a hypothesis on a weight function $\rho : \mathcal{S} \to (0, \infty)$ that plays an important role in the construction of a measure.

Assumption 3.19 Let $\mathcal{C} : \mathcal{S} \to (0, \infty)$ be a gentle function, and let $\beta > 0$. A weight function $\rho : \mathcal{S} \to (0, 1)$ is said to be $(\beta, \mathcal{C})$-*compatible* if there exists $K_2 \geq 1$ such that for all $B \in \mathcal{S}_m$, and $n > m$,

$$K_2^{-1}\pi(B)^{\beta}\mathcal{C}(B) \leq \sum_{B' \in \mathcal{D}_n(B)} \pi(B')^{\beta}\mathcal{C}(B') \leq K_2\pi(B)^{\beta}\mathcal{C}(B),$$

where $\mathcal{D}_n(B)$ denotes the descendants of B of generation n as defined in (3.12).

The above assumption is similar to (H4) in [23]. The following lemma is an analogue of [23, Lemma 2.7].

Lemma 3.20 *Let $(\mathcal{S}, D_{\mathcal{S}})$ be a hyperbolic filling of a doubling, K_P-uniformly perfect, compact metric space (X, d) as given in Lemma 3.9-(a). Let $\rho : \mathcal{S} \to (0, 1)$ be a weight function that satisfies (H1) and (H2). Let $\mathcal{C} : \mathcal{S} \to (0, \infty)$ be a gentle function, and let $\beta > 0$, such that ρ is $(\beta, \mathcal{S})$-compatible. For $n \geq 0$, denote*

$$\mu_n := \sum_{B \in \mathcal{S}_n} \pi(B)^{\beta}\mathcal{C}(B)\delta_{x_B},$$

where δ_{x_B} denotes the Dirac measure at x_B. Let μ be any weak subsequential limit of μ_n. Then there exists $C_1 > 1$ such that, for all $x \in X, r \leq \operatorname{diam}(X, d)/2$, and for all $B \in \mathcal{A}_{\mathcal{S}}(B(x, r))$, we have*

$$C_1^{-1}\pi(B)^{\beta}\mathcal{C}(B) \leq \mu((B(x, r)) \leq C_1\pi(B)^{\beta}\mathcal{C}(B), \tag{3.40}$$

where $\mathcal{A}_{\mathcal{S}}$ is as given in Definition 3.16. Furthermore, μ satisfies VD on (X, d).

Proof (Sketch of the proof) We only sketch the proof and skip the details as it follows from almost the same argument as [23, Lemma 2.7].

Let $x \in X, r \leq \operatorname{diam}(X)/2$, $B \in \mathcal{A}_{\mathcal{S}}(B(x, r))$ and $B = B(x_B, 2a^{-m})$. Choose $B_1 \in \mathcal{S}_{m+2}$ such that $x \in B_1 = B(x_{B_1}, 2a^{-m-2})$. By [23, (2.10)], the centers of all the descendants of B_1 belong to $\overline{B}(x, r/2)$. This along with $(\beta, \mathcal{C})$-compatibility implies that

$$\begin{aligned} \mu(B(x, r)) &\geq \mu(\overline{B}(x, r/2)) \\ &\geq \liminf_{n \to \infty} \mu_n\left(\{x_C : C \in \mathcal{D}_n(B_1)\}\right) \end{aligned}$$

$$
\begin{aligned}
&\quad (\text{since } \overline{B}(x, r/2) \supset \{x_C : C \in \mathcal{D}_n(B_1)\}) \\
&\gtrsim \liminf_{n \to \infty} \sum_{B' \in \mathcal{D}_n(B_1)} \pi(B')^\beta \mathcal{C}(B') \\
&\gtrsim \pi(B_1)^\beta \mathcal{C}(B_1) \gtrsim \pi(B)^\beta \mathcal{C}(B) \\
&\quad (\text{by } (\beta, \mathcal{C})\text{-compatibility and gentleness of } \mathcal{C}).
\end{aligned}
$$

By the argument in [23, proof of Lemma 2.7], for any $B' \in \mathcal{S}_n, n \geq m$ satisfying $x_{B'} \in B(x, r + a^{-m})$, we have $g(B')_n \sim g(B_1)_m$, where B_1 is as defined above. For the upper bound, for any $B' \in \mathcal{S}_n, n \geq m$ such that $x_{B'} \in B(x, r + a^{-m})$, we have that $g(B')_m \sim B$. Therefore, we estimate

$$
\begin{aligned}
\mu(B(x, r)) &\leq \mu(B(x, r + a^{-m})) \\
&\leq \limsup_{n \to \infty} \mu_n(B(x, r + a^{-m})) \\
&\leq \sum_{C \sim g(B_1)_m} \sum_{B' \in \mathcal{D}_n(C)} \pi(B')^\beta \mathcal{C}(B') \\
&\lesssim \sum_{C \sim g(B_1)_m} \pi(C)^\beta \mathcal{C}(C).
\end{aligned}
$$

Since $C \sim g(B_1)_m$ and $B \sim g(B_1)_m$, by gentleness of $\mathcal{C}$, we have $\mathcal{C}(C) \asymp \mathcal{C}(B)$. By (H2), we have $\pi(C) \asymp \pi(B)$. Furthermore, by doubling the number of such $C \in \mathcal{S}_m$ such that $C \sim g(B_1)_m$ is bounded by a constant that depends only on the parameters of the filling. Combining these estimates, we obtain the desired upper bound $\mu(B(x, r)) \lesssim \pi(B)^\beta \mathcal{C}(B)$. This completes the proof of (3.40).

The conclusion that μ satisfies VD follows from (3.40) and the gentleness of $\mathcal{C}$. $\square$

In the following proposition, we express the measure in Lemma 3.20 using the metric in Theorem 3.14-(a).

Proposition 3.21 *Let (X, d) be a compact, doubling, uniformly perfect metric space. Let $(\mathcal{S}, D_{\mathcal{S}})$ be a hyperbolic filling with parameters λ, a satisfying (3.13), (3.10) as given in Lemma 3.9-(a). Let $\mathcal{C} : \mathcal{S} \to (0, \infty)$ be a gentle function and let $\beta > 0$. Let $\rho : \mathcal{C} \to (0, 1)$ be a weight function that satisfies (H1), (H2), (H3), and $(\beta, \mathcal{C})$-compatibility. Let $\theta = \theta_\rho \in \mathcal{J}(X, d)$ denote the metric in Theorem 3.14-(a) and μ denote the measure on X constructed in Lemma 3.20. Then, there exist $C_1 > 1$ such that*

$$
C_1^{-1} r^\beta \mathcal{C}(B) \leq \mu(B_\theta(x, r)) \leq C_1 r^\beta \mathcal{C}(B), \tag{3.41}
$$

for all $x \in X, r < \operatorname{diam}(X,\theta)$, $B \in \mathcal{A}_{\mathcal{S}}(B_d(x,s))$, *where* s *is the largest number in* $[0, 2\operatorname{diam}(X,d)]$ *such that* $B_d(x,s) \subset B_\theta(x,r)$ *(as defined in* (3.4)*) and* $\mathcal{A}_{\mathcal{S}}(B_d(x,s))$ *is as given in Definition* 3.16.

Proof By an easy covering argument using the metric doubling property, it suffices to consider the case $r < \operatorname{diam}(X,\theta)/2$, so that $B_\theta(x,r) \neq X$.

Let $x \in X$, $0 < r < \operatorname{diam}(X,\theta)/2$ and let $s = \sup\{s_1 > 0 : B_d(x,s_1) \subset B_\theta(x,r)\}$. By Lemma 3.20, μ satisfies VD in (X,d). By (3.3) and in (3.39), μ satisfies VD in (X,θ) and there exists $C_2 > 1$ such that

$$C_2^{-1}\mu(B_d(x,s)) \leq \mu(B_\theta(x,r)) \leq C_2\mu(B_d(x,s)). \tag{3.42}$$

By (3.3), there exists $A_1 > 1$ such that $B_\theta(x,r) \subset A_1 \cdot B$ and $B \subset B_\theta(x, A_1 r)$ for all $B \in \mathcal{A}_{\mathcal{S}}(B_d(x,s))$. Hence by (3.7) and uniform perfectness, there exists $C_3 > 1$ such that

$$C_3^{-1} r \leq \operatorname{diam}(B,\theta) \leq C_3 r, \quad \text{for all } B \in \mathcal{A}_{\mathcal{S}}(B_d(x,s)). \tag{3.43}$$

By (3.34), (3.42), and (3.43), we obtain (3.41). □

3.6 Simplified hypotheses for construction of metric and measure

The goal of this section is to present an analogue of [23, Theorem 1.2] that will be used in the construction of metric measure space. Some of the main ideas in the proof of [23, Theorem 1.2] are inspired by the 'weight-loss program' of Keith and Laakso [58, §5.2].

We continue to consider a compact, doubling, uniformly perfect metric space (X,d), and a hyperbolic filling $\mathcal{S}_d = (\mathcal{S}, D_{\mathcal{S}})$ with horizontal parameter $\lambda \geq 8$ and vertical parameter $a > 1$ that satisfy (3.10). We consider $\beta > 0$, $\mathcal{C} : \mathcal{S} \to (0,\infty)$ such that $\mathcal{C}$ is gentle. Theorem 3.24 provides simpler sufficient conditions (S1), (S2) that allows us to construct a weight function that satisfies (H1), (H2), (H3), and is $(\beta,\mathcal{C})$-compatible. To state the sufficient conditions, we recall the following definition.

Definition 3.22 For $B \in \mathcal{S}_k$, $k \geq 0$, we define $\Gamma_{k+1}(B)$ as the set of horizontal paths $\gamma = \{B_i\}_{i=1}^N$, $N \geq 2$ such that $B_i \in \mathcal{S}_{k+1}$ for all $i = 1,\dots,N$, $B_i \sim B_{i+1}$ for all $i = 1,\dots,N-1$, $x_{B_1} \in B$, $x_{B_N} \notin 2\cdot B$, and $x_{B_i} \in 2\cdot B$ for all $i = 1,\dots,N-1$.

We introduce a subadditive estimate based on [11, Proposition 3.15].

Definition 3.23 We say that $B \in \mathcal{S}_k, k \geq 1$ is *non-peripheral* if every horizontal neighbour of B descends from the same parent. More precisely,

$B \in \mathcal{S}_k, k \geq 1$ is non-peripheral if

$$B \sim B' \text{ implies that } g(B)_{k-1} = g(B')_{k-1}.$$

By $\mathcal{N}$ we denote the set of all non-peripheral vertices in $\mathcal{S}$. We say that a function $\mathcal{C} : \mathcal{S} \to (0, \infty)$ satisfies (E) if it obeys the following estimate:

(E) there exists $\delta \in (0, 1)$ such that

$$\mathcal{C}(B) \leq (1 - \delta) \sum_{B' \in \mathcal{N} \cap \mathcal{D}_{k+1}(B)} \mathcal{C}(B')$$

for all $B \in \mathcal{S}_k, k \geq 1$.

In particular, the condition (E) implies $\mathcal{N} \cap \mathcal{D}_{k+1}(B) \neq \emptyset$ for all $k \geq 1$, $B \in \mathcal{S}_k$.

The following result is an analogue of [23, Theorem 1.2].

Theorem 3.24 *Let (X, d) be a compact, K_D-doubling, K_P-uniformly perfect metric space and let $\beta > 0$. Consider a hyperbolic filling $\mathcal{S}_d = (\mathcal{S}, D_{\mathcal{S}})$ with horizontal parameter $\lambda \geq 8$ and vertical parameter $a > 1$ that satisfies (3.10). Let $\mathcal{C} : \mathcal{S} \to (0, \infty)$ be a (K_h, K_v)-gentle function that satisfies (E). Then, there exists $\eta_0 \in (0, 1)$ that depends only on β, K_D, K_h, λ (but not on the vertical constants a, K_v or uniform perfectness constant K_P) such that the following is true. If there exists a function $\sigma : \mathcal{S} \to \left[0, \frac{1}{4}\right)$ that satisfies:*

(S1) *for all $B \in \mathcal{S}_k, k \geq 0$, if $\gamma = \{B_i : 1 \leq i \leq N\}$ is a path in $\Gamma_{k+1}(B)$ (as given in Definition 3.22), then*

$$\sum_{i=1}^{N} \sigma(B_i) \geq 1,$$

(S2) *and for all $k \geq 0$, and all $B \in \mathcal{S}_k$, we have*

$$\sum_{B' \in \mathcal{D}_{k+1}(B)} \sigma(B')^{\beta} \mathcal{C}(B') \leq \eta_0 \mathcal{C}(B),$$

then there exists a weight function $\rho : \mathcal{S} \to (0, 1)$ that satisfies (H1), (H2), (H3), and is $(\beta, \mathcal{C})$-compatible.

We recall some results from [23] that goes into the proof of Theorem 3.24. Let $\rho : \mathcal{S} \to (0, \infty)$ be a function, we define $\rho^* : \mathcal{S} \to (0, \infty)$ as

$$\rho^*(B) = \min_{B' \sim B} \rho(B), \quad \text{for } B \in \mathcal{S}.$$

We recall that $B \sim B'$ if there exists $k \geq 0$ such that $B, B' \in \mathcal{S}_k$ and $(\lambda \cdot B) \cap (\lambda \cdot B') \neq \emptyset$. If $\gamma = \{B_i\}_{i=1}^N$ is a horizontal path, we define

$$L_h(\gamma, \rho) = \sum_{j=1}^{N-1} \rho^*(B_j) \wedge \rho^*(B_{j+1}).$$

Proposition 3.25 ([23, Proposition 2.9]) *Let (X, d) be a compact, doubling, and uniformly perfect metric space. Let $\mathcal{S}$ be a hyperbolic filling with parameters a and λ satisfying (3.13). Assume that $\rho : \mathcal{S} \to (0, 1)$ satisfies (H1), (H2), and also the condition*

(H3') *for all $k \geq 1$, all $B \in \mathcal{S}_k$ and all $\gamma \in \Gamma_{k+1}(B)$, it holds that $L_h(\gamma, \rho) \geq 1$.*

Then the function ρ also satisfies (H3).

[23, Propostion 2.9] also assumes an additional assumption (H4) which was not used in the proof. In [23], the condition (H3') was stated for $k \geq 0$ but it is equivalent to the above condition because $\Gamma_1(B) = \emptyset$ for $B \in \mathcal{S}_0$.

Lemma 3.26 ([23, Lemma 2.13]) *Suppose we have a function $\pi_0 : \mathcal{S}_k \to (0, \infty)$ such that*

$$\forall B \sim B' \in \mathcal{S}_k, \quad \frac{1}{K} \leq \frac{\pi_0(B)}{\pi_0(B')} \leq K,$$

where $K \geq 1$ is a constant. Suppose that we have a function $\pi_1 : \mathcal{S}_{k+1} \to (0, \infty)$ which satisfies the following property:

$$\forall B \in \mathcal{S}_{k+1}, \exists A \in \mathcal{S}_k \text{ with } d(x_B, x_A) \leq 4a^{-k} \text{ and } 1 \leq \frac{\pi_0(A)}{\pi_1(B)} \leq K.$$

Define $\hat{\pi}_1 : \mathcal{S}_{k+1} \to (0, \infty)$ as

$$\hat{\pi}_1(B') = \pi_1(B') \vee \left(\frac{1}{K} \max \left\{ \pi_1(B) : B \sim B' \right\} \right).$$

Then, for all $B \sim B' \in \mathcal{S}_{k+1}$, we have

$$\frac{1}{K} \leq \frac{\hat{\pi}_1(B)}{\hat{\pi}_1(B')} \leq K.$$

The following is a slight modification of [23, Lemma 2.14].

Lemma 3.27 (Cf. [23, Lemma 2.14]) *Let $G = (V, E)$ be a graph whose vertices have degree bounded by D. Let $\mathcal{C} : V \to (0, \infty)$ be K-gentle; that is,*

$$K^{-1} \le \frac{\mathcal{C}(z)}{\mathcal{C}(z')} \le K, \text{ whenever there is an edge between } z \text{ and } z'.$$

Let Γ be a family of paths in G and let $\beta > 0$. Suppose that $\tau : V \to (0, \infty)$ is a function satisfying

$$\sum_{i=1}^{N-1} \tau(z_1) \ge 1, \text{ for all paths } \gamma = \{z_i\}_{i=1}^{N} \in \Gamma. \tag{3.44}$$

Define $\tilde{\tau} : V \to (0, \infty)$ as

$$\hat{\tau}(x) = 2 \max \{\tau(y) : y \in V_2(x)\},$$

where $V_2(x)$ denotes the set of all vertices whose graph distance from x is less than or equal to 2. Then $\tilde{\tau}$ satisfies

$$\sum_{i=1}^{N-1} \tilde{\tau}^*(z_i) \wedge \tilde{\tau}^*(z_{i+1}) \ge 1 \text{ for all paths } \gamma = \{z_i\}_{i=1}^{N} \in \Gamma,$$

where $\tilde{\tau}^*(x) = \min\{\tilde{\tau}(y) : y \sim x\}$, and such that

$$\sum_{z \in V} \tilde{\tau}(z)^{\beta} \mathcal{C}(z) \le 2^{\beta} D^2 K^2 \sum_{z \in V} \tau(z)^{\beta} \mathcal{C}(z). \tag{3.45}$$

Proof As shown in [23, Lemma 2.14] the function $\tilde{\tau}$ satisfies (3.44).

Since $\mathcal{C}$ is K-gentle and $\sup_{x \in V} |V_2(x)| \le D^2$, we obtain

$$\begin{aligned}
\sum_{x \in V} \tilde{\tau}(x)^{\beta} \mathcal{C}(x) &\le 2^{\beta} \sum_{x \in V} \sum_{z \in V_2(x)} \tau(z)^{\beta} \mathcal{C}(x) \\
&\le 2^{\beta} K^2 \sum_{x \in V} \sum_{z \in V_2(x)} \tau(z)^{\beta} \mathcal{C}(z) = 2^{\beta} \sum_{z \in X} \sum_{x \in V_2(z)} \tau(z)^{\beta} \mathcal{C}(z) \\
&\le 2^{\beta} K^2 D^2 \sum_{z \in X} \tau(z)^{\beta} \mathcal{C}(z).
\end{aligned}$$

□

Proof of Theorem 3.24 Let D_h be such that

$$D_h \geq \sup_{k\geq 0} \max_{B\in\mathcal{S}_k} \left|\left\{B' \in \mathcal{S}_k : B' \sim B\right\}\right| \tag{3.46}$$

By K_D-doubling, D_h and can be chosen to depend only on λ and K_D [40, Exercise 10.17]. Similarly, the number of children can be bounded by a constant D_v that depends only on a and K_D with

$$D_v \geq \sup_{k\geq 0} \max_{B\in\mathcal{S}_k} |\mathcal{D}_{k+1}(B)|. \tag{3.47}$$

Take $\eta_0 \in (0, 1)$ be a constant which will be fixed later, and set

$$\eta_- := \left(\eta_0 K_v^{-1} D_v^{-1}\right)^{1/\beta} \wedge \frac{1}{4}.$$

Let $\sigma : \mathcal{S} \to [0, \frac{1}{4})$ satisfy (S1) and (S2). Define $\tau = \sigma \vee \eta_-$. Then

$$\begin{aligned}\sum_{B'\in\mathcal{D}_{k+1}(B)} \tau(B')^\beta \mathcal{C}(B') &\leq \sum_{B'\in\mathcal{D}_{k+1}(B)} \sigma(B')^\beta \mathcal{C}(B') + \eta_-^\beta D_v K_v \mathcal{C}(B)\\ &\leq 2\eta_0 \mathcal{C}(B).\end{aligned}$$

For $B \in \mathcal{S}_k$ define $V_{2,k}(B) = \left\{B' \in \mathcal{S}_k : \exists B'' \in \mathcal{S}_k \text{ such that } B \sim B'' \sim B\right\}$. Then by Lemma 3.27, the function

$$\tilde{\tau}(B) = 2 \max\left\{\tau(B') : B' \in V_{2,k}(B)\right\}, \quad \text{for all } B \in \mathcal{S}_k$$

satisfies (H3') and

$$\begin{aligned}\sum_{B'\in\mathcal{D}_{k+1}(B)} \tilde{\tau}(B')^\beta \mathcal{C}(B') &\leq 2^\beta K_h^2 D_h^2 \sum_{B'\in\mathcal{D}_{k+1}(B)} \tau(B')^\beta \mathcal{C}(B')\\ &\leq 2^{\beta+1} K_h^2 D_h^2 \eta_0 \mathcal{C}(B),\end{aligned} \tag{3.48}$$

for all $B \in \mathcal{S}_k$.

We construct $\hat{\rho} : \mathcal{S} \to (0, 1)$ satisfying

(1) $\hat{\rho} \geq \tilde{\tau}$. In particular $\hat{\rho}$ satisfies (H3') and $\hat{\rho}(B) \geq \eta_-$ for all $B \in \mathcal{S}$.
(2) (H2) with constant K, where $K = \eta_-^{-1}$.
(3) $\hat{\rho}(B) \leq \max\left\{\tilde{\tau}(B') : B' \sim B\right\}$. In particular, $\hat{\rho}(B) \leq \frac{1}{2}$ for all $B \in \mathcal{S}$.

We briefly recall the construction in [23]. Set $\hat{\rho}(w) = \frac{1}{2}$, where $w \in \mathcal{S}_0$. Note that $\tilde{\tau} \leq \frac{1}{2} \leq 1$ (since $\eta_- \leq \frac{1}{2}$ and $\sigma \leq \frac{1}{4}$). We construct $\hat{\rho}$ inductively on $\mathcal{S}_k$.

Suppose we have constructed $\hat{\rho}_i$ for $i = 1, \ldots, k$. We construct $\hat{\rho}_{k+1}$ using Lemma 3.26. We denote

$$\pi_0(A) = \prod_{i=0}^{k} \hat{\rho}_i(g(A)_i) \text{ for } A \in \mathcal{S}_k \text{ and,}$$
$$\pi_1(B) = \tilde{\tau}(B)\pi_0(g(B)_j) \text{ for } B \in \mathcal{S}_{k+1}.$$

By the induction hypothesis along with Lemma 3.26, we obtain a function $\hat{\pi}_1 : \mathcal{S}_{j+1} \to (0, \infty)$ that satisfies $K^{-1}\hat{\pi}_1(B') \leq \hat{\pi}_1(B) \leq K\hat{\pi}_1(B')$ for all $B \sim B' \in \mathcal{S}_{j+1}$. We define $\hat{\rho} : \mathcal{S}_{j+1} \to (0, \infty)$ as

$$\hat{\rho}_{k+1}(B) = \frac{\hat{\pi}_1(B)}{\pi_0(g(B)_j)}.$$

Carrasco Piaggio's proof of [23, Theorem 1.2] shows that $\hat{\rho}$ satisfies properties (1), (2), and (3) above. For any $B \in \mathcal{S}_k$, $k \geq 0$, using (3.48) we estimate

$$\begin{aligned}
&\sum_{B' \in \mathcal{D}_{k+1}} \hat{\rho}(B')^{\beta} \mathcal{C}(B') \\
&\leq \sum_{B' \in \mathcal{D}_{k+1}(B)} \sum_{B'' \sim B'} \tilde{\tau}(B'')^{\beta} \mathcal{C}(B') \text{ (by property (3) above)} \\
&\leq K_h \sum_{B' \in \mathcal{D}_{k+1}(B)} \sum_{B'' \sim B'} \tilde{\tau}(B'')^{\beta} \mathcal{C}(B'') \\
&\leq K_h D_h \sum_{C \sim B} \sum_{B'' \in \mathcal{D}_{k+1}(C)} \tilde{\tau}(B'')^{\beta} \mathcal{C}(B'') (\because B'' \sim B' \text{ implies } g(B'')_k \sim B) \\
&\leq 2^{\beta+1} K_h^3 D_h^3 \eta_0 \sum_{C \sim B} \mathcal{C}(C) \leq 2^{\beta+1} K_h^4 D_h^4 \eta_0 \mathcal{C}(B).
\end{aligned} \tag{3.49}$$

Now choose η_0

$$2^{\beta+1} K_h^3 D_h^3 \eta_0 = \frac{1}{2}, \tag{3.50}$$

so that (3.49) yields

$$\sum_{B' \in \mathcal{D}_{k+1}} \hat{\rho}(B')^{\beta} \mathcal{C}(B') \leq \frac{1}{2} \mathcal{C}(B) \quad \text{for all } B \in \mathcal{S}_k, k \geq 0. \tag{3.51}$$

Note from (3.50) that η_0 depends only on β, K_h, K_D, λ but not on constants K_v, a, K_P.

Next, we modify $\hat{\rho}$ so that it becomes $(\beta, \mathcal{C})$-compatible. For each $B \in \mathcal{S}_k$, $k \geq 0$, we choose $\omega_B \geq 0$ such that

$$\rho(B') = \begin{cases} \omega_B \vee \hat{\rho}(B') & \text{if } B' \in \mathcal{D}_{k+1}(B) \cap \mathcal{N}, \\ \hat{\rho}(B') & \text{if } B' \in \mathcal{D}_{k+1}(B) \setminus \mathcal{N} \end{cases} \tag{3.52}$$

satisfies

$$\sum_{B' \in \mathcal{D}_{k+1}(B)} \rho(B')^\beta \mathcal{C}(B') = \mathcal{C}(B). \tag{3.53}$$

The existence of an $\omega_B \in (0, \infty)$ that satisfies (3.53) follows from the intermediate value theorem. In particular, we use (3.51), the continuity of the map

$$\omega_B \mapsto \sum_{B' \in \mathcal{D}_{k+1} \cap \mathcal{N}} (\omega_B \vee \rho(B'))^\beta \mathcal{C}(B') + \sum_{B' \in \mathcal{D}_{k+1} \setminus \mathcal{N}} \hat{\rho}(B')^\beta \mathcal{C}(B'),$$

along with the fact that $\mathcal{D}_{k+1} \cap \mathcal{N}$ is non-empty. The equality (3.53) implies that ρ is $(\beta, \mathcal{C})$-compatible since

$$\sum_{B' \in \mathcal{D}_n(B)} \pi(B')^\beta \mathcal{C}(B') = \pi(B)^\beta \mathcal{C}(B)$$

for all $B \in \mathcal{S}_k$ and for all $n \geq k$.

It remains to show that ρ satisfies (H1), (H2) and (H3). We start by verifying (H1). Clearly $\rho(B) \geq \hat{\rho}(B) \geq \eta_-$ for all $B \in \mathcal{S}_k$. On the other hand, (E) implies that $\omega_B \leq (1-\delta)^{1/\beta}$, since

$$\begin{aligned} \sum_{B' \in \mathcal{D}_{k+1}(B) \cap \mathcal{N}} \omega_B^\beta \mathcal{C}(B') &\leq \sum_{B' \in \mathcal{D}_{k+1}(B)} \rho(B')^\beta \mathcal{C}(B') = \mathcal{C}(B) \\ &\leq (1-\delta) \sum_{B' \in \mathcal{D}_{k+1}(B) \cap \mathcal{N}} \mathcal{C}(B'). \end{aligned}$$

This combined with $\sigma \leq \frac{1}{4}$ and property (3) of $\hat{\rho}$ implies that

$$\eta_- \leq \rho(B) \leq (1/2) \vee (1-\delta)^{1/\beta}.$$

By setting $\eta_+ = (1/2) \vee (1-\delta)^{1/\beta} \in (0, 1)$, we obtain (H1).

Since $\rho \geq \hat{\rho}$, ρ satisfies (H3'). Therefore by Proposition 3.25 it suffices to show (H2). Let $B \sim B' \in \mathcal{S}_k$, $k \geq 1$. We consider two cases.

Case 1: $g(B)_{k-1} = g(B')_{k-1}$. Then $\frac{\pi(B)}{\pi(B')} = \frac{\rho(B)}{\rho(B')}$ which thanks to (H1) satisfies

$$\eta_- \le \frac{\pi(B)}{\pi(B')} \le \eta_-^{-1}.$$

Case 2: $g(B)_{k-1} \neq g(B')_{k-1}$. Let $n \ge 0$ be the maximal integer such that $g(B)_n = g(B')_n$. In this case for $i = n+1, \ldots, k$, we have $g(B)_i \sim g(B')_i$. Hence for $i = n+2, \ldots, k$, $g(B)_i$ and $g(B')_i$ must both be peripheral (belong to $\mathcal{N}^c$). Therefore

$$\begin{aligned}\frac{\pi(B)}{\pi(B')} &= \frac{\rho(g(B)_{n+1})}{\rho(g(B')_{n+1})} \prod_{i=n+2}^{k} \frac{\rho(g(B)_i)}{\rho(g(B')_i)} = \frac{\rho(g(B)_{n+1})}{\rho(g(B')_{n+1})} \prod_{i=n+2}^{k} \frac{\hat\rho(g(B)_i)}{\hat\rho(g(B')_i)} \\ &= \frac{\rho(g(B)_{n+1})}{\rho(g(B')_{n+1})} \frac{\hat\pi(B)}{\hat\pi(B')} \frac{\hat\rho(g(B')_{n+1})}{\hat\rho(g(B)_{n+1})}.\end{aligned}$$

By combining property (2) of $\hat\rho$ to estimate $\frac{\hat\pi(B)}{\hat\pi(B')}$ and $\eta_- \le \hat\rho \le \rho \le 1$ for the remaining terms, we obtain

$$\eta_-^3 \le \frac{\pi(B)}{\pi(B')} \le \eta_-^{-3}.$$

Combining the two cases, we obtain (H2) with constant $K_0 = \eta_-^{-3}$. □

Remark 3.28 One of the key differences between the construction in [23] and our work is the proof of Theorem 3.24. In the construction in [23], a similar modification as defined in (3.52) was done but $\mathcal{N}$ was chosen to be a singleton set. However, that choice does not work in our context because we need to ensure that $\omega_B \le \eta_+$, where $\eta_+ \in (0, 1)$. This is because $\mathcal{C}(B')$ can be strictly smaller than $\mathcal{C}(B)$. The construction in [23] can be interpreted as the particular case $\mathcal{C}(B) = 1$ for all $B \in \mathcal{S}$. The requirement $\eta_+ \in (0, 1)$ is the motivation behind the notion of non-peripheral vertices and the enhanced subadditive estimate (E).

The following ‘patching lemma’ allows us to combine functions that satisfy local versions of (S1) and (S2) into a global one. This is an adaptation of the construction in [23, pp. 533–534].

Lemma 3.29 (Patching lemma) *Let $\mathcal{S}$ denote a hyperbolic filling of a K_D-doubling, uniformly perfect, compact metric space, and let $\beta, \eta_1 > 0$. Let $\mathcal{S}_d = (\mathcal{S}, D_{\mathcal{S}})$ be a hyperbolic filling with horizontal parameter $\lambda \ge 8$ and vertical parameter $a > 1$ that satisfies* (3.10). *Let $\mathcal{C} : \mathcal{S} \to (0, \infty)$ be a (K_h, K_v)-gentle function. Assume that for all $B \in \mathcal{S}_k$, $k \ge 1$, there exists $\sigma_B : \mathcal{S}_{k+1} \to [0, \frac{1}{4})$ such that* *(a) if we set* $V_B = \left\{B' \in \mathcal{S}_{k+1} : B' \cap 3 \cdot B \neq \emptyset\right\}$*, then* $\sigma_B(B') = 0$ *for all* $B' \in \mathcal{S}_{k+1} \setminus V_B$*.*

(b) for any path $\gamma = \{B_i\}_{i=1}^N \in \Gamma_{k+1}(B)$*, we have*

$$\sum_{i=1}^{N} \sigma_B(B_i) \geq 1,$$

(c) and $\sum_{B' \in \mathcal{S}_{k+1}} \sigma_B(B')^\beta \mathcal{C}(B') \leq \eta_1 \mathcal{C}(B)$*.*

Let $\sigma : \mathcal{S} \to [0, \frac{1}{4})$ *be defined as*

$$\sigma(B') = \max\left\{\sigma_A(B') : A \in \mathcal{S}_k\right\}$$

for all $B' \in \mathcal{S}_{k+1}$ *and for all* $k \geq 1$*, and* $\sigma(B') = 0$ *for all* $B' \in \mathcal{S}_0 \cup \mathcal{S}_1$*. Then there exists* $C_{3.29} \geq 1$ *that depends only on* K_D*,* K_h *such that* σ *satisfies* (S1) *and the estimate*

$$\sum_{B' \in \mathcal{D}_{k+1}(B)} \sigma(B')^\beta \mathcal{C}(B') \leq C_{3.29} \eta_1 \mathcal{C}(B).$$

Proof For any path $\gamma = \{B_i\}_{i=1}^N \in \Gamma_{k+1}(B)$, $B \in \mathcal{S}_k$, we have $\sum_{i=1}^N \sigma(B_i) \geq \sum_{i=1}^N \sigma_B(B_i) \geq 1$. Therefore σ satisfies (S1).

For any $B \in \mathcal{S}_{k+1}$, we have

$$\begin{aligned}
\sum_{B' \in \mathcal{D}_{k+1}(B)} \sigma(B')^\beta \mathcal{C}(B') &= \sum_{B' \in \mathcal{D}_{k+1}(B)} \max\left\{\sigma_A(B')^\beta : A \in \mathcal{S}_k\right\} \mathcal{C}(B') \\
&\leq \sum_{B' \in \mathcal{D}_{k+1}(B)} \sum_{A : B' \in V_A} \sigma_A(B')^\beta \mathcal{C}(B') \\
&\leq \sum_{A : V_B \cap V_A \neq \emptyset} \sum_{B' \in V_A} \sigma_A(B')^\beta \mathcal{C}(B') \\
&\leq \sum_{A : V_B \cap V_A \neq \emptyset} \eta_1 \mathcal{C}(A) \\
&\leq \sum_{A : A \sim B} \eta_1 \mathcal{C}(A) \quad (\because V_B \cap V_A \neq \emptyset \implies A \sim B) \\
&\leq \sum_{A : A \sim B} \eta_1 K_h \mathcal{C}(B) \leq D_h K_h \eta_1 \mathcal{C}(B),
\end{aligned}$$

where D_h is chosen as (3.46). □

4 Universality of the conformal walk dimension

4.1 Consequences of Harnack inequalities

In this subsection, we recall some previous results concerning the elliptic and parabolic Harnack inequalities. We start with recalling the definition of the heat kernel and its sub-Gaussian estimates.

Definition 4.1 (HKE(β)) Let $(X, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space, and let $\{P_t\}_{t>0}$ denote its associated Markov semigroup. A family $\{p_t\}_{t>0}$ of non-negative Borel measurable functions on $X \times X$ is called the *heat kernel* of $(X, d, m, \mathcal{E}, \mathcal{F})$, if p_t is the integral kernel of the operator P_t for any $t > 0$, that is, for any $t > 0$ and for any $f \in L^2(X, m)$,

$$P_t f(x) = \int_X p_t(x, y) f(y)\, dm(y) \qquad \text{for } m\text{-a.e. } x \in X.$$

Let $\beta \in (1, \infty)$. We say that $(X, d, m, \mathcal{E}, \mathcal{F})$ satisfies the *heat kernel estimates* HKE(β) with walk dimension β, if there exist $C_1, c_1, c_2, \delta \in (0, \infty)$ and a heat kernel $\{p_t\}_{t>0}$ such that for any $t > 0$,

$$p_t(x, y) \le \frac{C_1}{m\big(B(x, t^{1/\beta})\big)} \exp\left(-c_1 \left(\frac{d(x, y)^\beta}{t}\right)^{\frac{1}{\beta-1}}\right) \qquad \text{for } m\text{-a.e. } x, y \in X, \tag{4.1}$$

$$p_t(x, y) \ge \frac{c_2}{m\big(B(x, t^{1/\beta})\big)} \qquad \text{for } m\text{-a.e. } x, y \in X \text{ with } d(x, y) \le \delta t^{1/\beta}. \tag{4.2}$$

The following condition is the key to establishing the parabolic Harnack inequality in the presence of the elliptic Harnack inequality.

Definition 4.2 (Capacity estimate) Let $\beta \in (1, \infty)$. We say that an MMD space $(X, d, m, \mathcal{E}, \mathcal{F})$ satisfies the *capacity estimate* cap(β) if there exist $C_1, A_1, A_2 > 1$ such that for all $R \in (0, \operatorname{diam}(X, d)/A_2)$, $x \in X$

$$C_1^{-1} \frac{m(B(x, R))}{R^\beta} \le \operatorname{Cap}(B(x, R), B(x, A_1 R)^c) \le C_1 \frac{m(B(x, R))}{R^\beta}. \tag{cap(β)}$$

Poincaré and cutoff Sobolev inequalties are important functional inequalties for obtaining the stability of Harnack inequalties, which we recall below.

Definition 4.3 Let $(X, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space and let $\beta \in (1, \infty)$.

(i) We say that $(X, d, m, \mathcal{E}, \mathcal{F})$ satisfies the *Poincaré inequality* PI(β), if there exist constants $C > 0$ and $A \ge 1$ such that for all $x \in X$, $R \in (0, \infty)$ and $f \in \mathcal{F}$,

$$\int_{B(x,R)} (f - f_{B(x,R)})^2 \, dm \le C R^{\beta} \int_{B(x,AR)} d\Gamma(f, f), \qquad \mathrm{PI}(\beta)$$

where $f_{B(x,R)} := \frac{1}{\mu(B(x,R))} \int_{B(x,R)} f \, dm$.

(ii) Let $B_1 \subset B_2$ be open subsets of X. We say that $\varphi \in \mathcal{F}$ is a *cutoff function* for $B_1 \subset B_2$ if $0 \le \varphi \le 1$ m-a.e., $\varphi \equiv 1$ m-a.e. on B_1 and $\mathrm{supp}_m[\varphi] \subset B_2$.

(iii) We say that $(X, d, m, \mathcal{E}, \mathcal{F})$ satisfies the *cutoff Sobolev inequality* $\mathrm{CS}(\beta)$, if there exist $C_1, C_2, C_3, \eta > 0$ and $A > 1$ such that the following holds. For all $x \in X$, $R > 0$ with $B_1 = B(x, R)$, $B_2 = B(x, AR)$, there exists a cutoff function $\varphi \in \mathcal{F} \cap C(X)$ for $B_1 \subset B_2$ such that for any $u \in \mathcal{F}$,

$$\int_X u^2 \, d\Gamma(\varphi, \varphi) \le C_1 \int_{B_2 \setminus B_1} \Gamma(u, u) + \frac{C_2}{R^{\beta}} \int_{B_2 \setminus B_1} u^2 \, dm, \qquad \mathrm{CS}(\beta)$$

and such that the following scale invariant Hölder continuity estimate holds:

$$|\varphi(x_1) - \varphi(x_2)| \le C_3 \left(\frac{d(x_1, x_2)}{R} \right)^{\eta} \tag{4.3}$$

for all $x_1, x_2 \in X$.

(iv) We say that $(X, d, m, \mathcal{E}, \mathcal{F})$ satisfies the *weak cutoff Sobolev inequality* $\mathrm{CS}_{\mathrm{weak}}(\beta)$, if $\mathrm{CS}(\beta)$ with "$\varphi \in \mathcal{F} \cap C(X)$" replaced by "$\varphi \in \mathcal{F}$" and with the Hölder continuity estimate (4.3) dropped holds.

The following lemma shows that, under the above Poincaré and cutoff Sobolev inequalities, the extended Dirichlet space $\mathcal{F}_e$ is contained in the space $\mathcal{F}_{\mathrm{loc}}$ as defined in (2.3) of functions locally in the domain $\mathcal{F}$ of the Dirichlet form $(\mathcal{E}, \mathcal{F})$.

Lemma 4.4 *Let $(X, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space that satisfies $\mathrm{PI}(\beta_1)$ and $\mathrm{CS}_{\mathrm{weak}}(\beta_2)$ for some $\beta_1, \beta_2 \in (1, \infty)$. Then $\mathcal{F}_e \subset \mathcal{F}_{\mathrm{loc}}$.*

Proof Let $g \in \mathcal{F}_e$. Then there exists an $\mathcal{E}$-Cauchy sequence $\{g_n\}_n \subset \mathcal{F}$ such that g_n converges to g m-a.e. Let $B = B(x, R)$ be any ball. By the Poincaré inequality $\mathrm{PI}(\beta_1)$ the sequence $g_n - (g_n)_B$ is $L^2(B, m)$-Cauchy. Since g_n converges to g m-a.e. and $g_n - (g_n)_B$ is $L^2(B, m)$-Cauchy, we have that $\lim_{n \to \infty} (g_n)_B = g_B$ and that g_n converges to g in $L^2(B, m)$.

Let $A > 1$ be as in $\mathrm{CS}_{\mathrm{weak}}(\beta_2)$ and let φ be a cutoff function for $B = B(x, R) \subset B(x, AR)$ as in $\mathrm{CS}_{\mathrm{weak}}(\beta_2)$. By [33, Theorem 2.1.7] we may assume that g_n is bounded, so that $g_n \varphi \in \mathcal{F}$ by [33, Theorem 1.4.2-(ii)]. Noting that $\varphi^2 \le 1$ q.e. by [33, Lemma 2.1.4], by the Leibniz rule [33, Lemma 3.2.5] and the Cauchy–Schwarz inequality [33, Lemma 5.6.1] for $\Gamma(\cdot, \cdot)$, we obtain

$$\frac{1}{2}\mathcal{E}(\varphi(g_n - g_m), \varphi(g_n - g_m)) \\ \leq \mathcal{E}(g_n - g_m, g_n - g_m) + \int_X (g_n - g_m)^2 \, d\Gamma(\varphi, \varphi),$$

which together with $\mathrm{CS}_{\mathrm{weak}}(\beta_2)$ for $u = g_n - g_m$ and the previous paragraph with $B(x, AR)$ in place of $B = B(x, R)$ shows that $g_n\varphi$ is a Cauchy sequence in $(\mathcal{F}, \mathcal{E}_1)$ converging in $L^2(X, m)$ to $g\varphi$. Thus $g\varphi \in \mathcal{F}$ by the completeness of $(\mathcal{F}, \mathcal{E}_1)$, and since $g\varphi = g$ in B and $B = B(x, R)$ is an arbitrary ball in (X, d), we conclude that $g \in \mathcal{F}_{\mathrm{loc}}$. □

We record the following theorem which relates the elliptic and parabolic Harnack inequalities. The equivalence of (a), (b) and (c) is due to Grigor'yan and Telcs [38, Theorem 3.1] in the context of random walks on graphs. This was later extended to the MMD space setting by several authors. The equivalence between (a) and (d) is due to Barlow and Bass [6] for random walks on graphs and was extended to the current setting in [7].

Theorem 4.5 *Let $(X, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space and let $\beta \in (1, \infty)$. Then the following are equivalent:*

(a) $(X, d, m, \mathcal{E}, \mathcal{F})$ satisfies PHI(β).
(b) $(X, d, m, \mathcal{E}, \mathcal{F})$ satisfies VD, EHI *and* cap(β).
(c) $(X, d, m, \mathcal{E}, \mathcal{F})$ satisfies VD *and* HKE(β).
(d) $(X, d, m, \mathcal{E}, \mathcal{F})$ satisfies VD, PI(β) *and* CS(β).
(e) $(X, d, m, \mathcal{E}, \mathcal{F})$ satisfies VD, PI(β) *and* $\mathrm{CS}_{\mathrm{weak}}(\beta)$.

Moreover, if $(X, d, m, \mathcal{E}, \mathcal{F})$ satisfies any one of (a), (b), (c), (d) *and* (e), *then (X, d) is arcwise connected and uniformly perfect and (X, d, m) is* RVD.

Proof First, by Remark 2.6, [36, Proposition 5.6] and [9, Lemma 5.2-(c),(b)] (see also [9, Proof of Theorem 5.4, (b) ⇒ (a)]), (b) implies that (X, d) is arcwise connected and uniformly perfect and that (X, d, m) is RVD. Then since (b) and RVD together imply (c) by [37, Theorem 1.2], it follows that (b) implies (c). Next, (c) implies (a) by [10, Theorem 3.1], CS(β) by [7, Section 3],[3] PI(β) by [37, Proof of Theorem 1.2] or [73, Proof of Theorem 3.2] (see also [56, Remark 2.9-(b)]), and thus (d). It is obvious that (d) implies (e). Conversely, since the conjunction of VD and PI(β) implies RVD by [80, Corollary 2.3] and Remark 3.18-(b), it follows from [37, Theorem 1.1] along with [11, Proposition 5.11 and Remark 5.12] that (e) implies (b); in [37, Theorem 1.1] the condition EHI is stated and proved only for $h \in \mathcal{F}$, but by using Lemma 4.4

[3] We note that the proof of the Hölder continuity estimate of the resolvent kernel stated in [7, Lemma 3.3], from which (4.3) follows, has a gap which has been resolved in the arXiv version. The proof of CS(β) in [7, Section 3] works also in the compact setting with minor modifications and it does not use the assumption that the metric d is geodesic.

and the relative compactness in X of all balls in (X, d) we obtain our version of EHI in Definition 2.5, where $h \in \mathcal{F}_e$.

It remains to prove that (a) implies (c). Since the conjunction of VD and PHI(β) implies HKE(β) by [10, Theorem 3.1] (see also [73, Proof of Theorem 3.2]), it suffices to show that PHI(β) implies VD.

The implication from PHI(β) to VD follows from [10, Theorem 3.2] under the additional assumption that the metric d is geodesic. However, this additional assumption is not necessary and we modify the proof in [10] as follows. By [10, Lemma 4.6], there exists a heat kernel $p_t(x, y)$ such that $(t, x, y) \mapsto p_t(x, y)$ is continuous on $(0, \infty) \times X \times X$. By [10, (4.52)], there exists $c_1, c_2 > 0$ such that

$$\sup_{x,y\in B(x_0,r)} p_t(x, y) \geq \frac{c_1}{m(B(x_0, r))} \exp\left(-\frac{c_2 t}{r^\beta}\right) \quad \text{for all } x_0 \in X, r > 0, t > 0. \tag{4.4}$$

Let $0 < C_1 < C_2 < C_3 < C_4$, $\delta \in (0, 1)$ and $C_5 > 1$ denote the constants in PHI(β). Define $K = \frac{C_3+C_4}{C_1+C_2} \in (1, \infty)$.

Let $x_0 \in X, r > 0$ be arbitrary. Fix $t > 0$ such that $t = (C_1 + C_2)\delta^{-\beta} r^\beta/2$. Using (4.4), we choose $y \in B(x_0, r)$ such that $\sup_{x\in B(x_0,r)} p_t(x, y) \geq \frac{1}{2} \frac{c_1}{2m(B(x_0,r))} \exp\left(-\frac{c_2 t}{r^\beta}\right)$. By PHI($\beta$), we obtain

$$p_{Kt}(x_0, y) \geq \frac{C_5^{-1} c_1}{2m(B(x_0, r))} \exp\left(-\frac{c_2 t}{r^\beta}\right) \quad \text{for some } y \in B(x_0, r). \tag{4.5}$$

By PHI(β) for the caloric function $(t, z) \mapsto p_t(x_0, z)$ on the cylinder $(0, C_5\delta^{-\beta} r_1^\beta) \times B(x, \delta^{-1} r_1)$, where $r_1 > 0$ satisfies $(C_1 + C_2)\delta^{-\beta} r_1^\beta/2 = Kt$ (or equivalently, $r_1 = K^{1/\beta} r$) and (4.5), we obtain

$$p_{K^2 t}(x_0, z) \geq \frac{C_5^{-2} c_1}{2m(B(x_0, r))} \exp\left(-\frac{c_2 t}{r^\beta}\right) \quad \text{for all } z \in B(x_0, K^{1/\beta} r). \tag{4.6}$$

Using $\int_X p_{K^2 t}(x_0, z)\, m(dz) \leq 1$ and $t = (C_1 + C_2)\delta^{-\beta} r^\beta/2$ and (4.6), there exists $C_6 > 1$ such that

$$\frac{m(B(x_0, K^{1/\beta} r))}{m(B(x_0, r))} \leq C_6, \quad \text{for all } x_0 \in X, r > 0.$$

By iterating the above estimate $\lceil \beta \log 2/ \log K \rceil$ times, we obtain the volume doubling property VD. $\square$

Remark 4.6 Theorem 4.5 can be generalized to the case where the space-time scaling function $\Psi(r) = r^{\beta}$ is replaced with a homeomorphism $\Psi : [0,\infty) \to [0,\infty)$ satisfying the following estimates: there exist $C_1, \beta_1, \beta_2 \in (0,\infty)$ with $1 < \beta_1 \leq \beta_2$ such that

$$C_1^{-1}\left(\frac{R}{r}\right)^{\beta_1} \leq \frac{\Psi(R)}{\Psi(r)} \leq C_1\left(\frac{R}{r}\right)^{\beta_2} \quad \text{for all } r, R \in (0,\infty) \text{ with } r \leq R.$$

The generalized version of the relevant properties like PHI(β) and cap(β) for such space-time scale functions can be found in [10,37].

Combining Theorem 4.5 with the main result of [80], we have the following alternative proof that PHI(β) cannot hold for $\beta \in (0,2)$ and thereby that (1.5) holds.

Lemma 4.7 *If $\beta > 0$ and an MMD space $(X, d, m, \mathcal{E}, \mathcal{F})$ satisfies* PHI(β), *then $\beta \geq 2$.*

Proof Assume to the contrary that $\beta < 2$, so that $\rho := d^{\beta/2}$ would be a metric on X and $(X, \rho, m, \mathcal{E}, \mathcal{F})$ would satisfy PHI(2). For $\varepsilon > 0$, we define the ε-chain metric as

$$d_{\varepsilon}(x, y) = \inf \sum_{i=0}^{N-1} d(x_i, x_{i+1}),$$

where the infimum is taken over all finite collection of points $\{x_i\}_{i=0}^{N} \subset X$, $N \in \mathbb{N}$, such that $x_0 = x$, $x_N = y$ and $d(x_i, x_{i+1}) < \varepsilon$ for all $i = 0, 1, \ldots, N-1$. We define $\rho_{\varepsilon}(x, y)$ analogously for any $x, y \in X$. Then since $(X, \rho, m, \mathcal{E}, \mathcal{F})$ would satisfy cap(2) and PI(2) by Theorem 4.5, it follows from [80, Theorem 1.6 and Remark 1.7(a)] that there would exist $C > 0$ such that

$$\rho_{\varepsilon}(x, y) \leq C\rho(x, y) \quad \text{for all } x, y \in X \text{ and } \varepsilon > 0. \tag{4.7}$$

On the other hand, since $d(x_i, x_{i+1}) < \varepsilon$ is equivalent to $\rho(x_i, x_{i+1}) \leq \varepsilon^{\beta/2}$, we would have

$$d(x, y) \leq d_{\varepsilon}(x, y) \leq \varepsilon^{1-\beta/2}\rho_{\varepsilon^{2/\beta}}(x, y) \quad \text{for all } x, y \in X \text{ and } \varepsilon > 0. \tag{4.8}$$

Combining (4.7) and (4.8), we would obtain

$$d(x, y) \leq C\varepsilon^{1-\beta/2}\rho(x, y) = C\varepsilon^{1-\beta/2}d(x, y)^{\beta/2}$$

for all $x, y \in X$ and $\varepsilon > 0$, and letting $\varepsilon \downarrow 0$ would yield $\mathrm{diam}(X, d) = 0$, which would contradict our standing assumption that X contains at least two elements. □

We next collect some properties of EHI in relation to time changes and quasisymmetric changes of the metric. The following lemma was observed first by Jun Kigami and taught through [65] to the authors of [11] and of the present paper.

Lemma 4.8 ([65], cf. [11, Lemma 5.3]) *Let $(X, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space, let $\mu \in \mathcal{A}(X, d, m, \mathcal{E}, \mathcal{F})$ and let $\theta \in \mathcal{J}(X, d)$. Then $(X, d, m, \mathcal{E}, \mathcal{F})$ satisfies* EHI *if and only if $(X, \theta, \mu, \mathcal{E}^{\mu}, \mathcal{F}^{\mu})$ satisfies* EHI.

Proof $((\mathcal{F}^{\mu})_e, \mathcal{E}^{\mu}) = (\mathcal{F}_e, \mathcal{E})$ by [24, Corollary 5.2.12], $\mathcal{F}^{\mu} \cap C_c(X) = \mathcal{F}_e \cap C_c(X) = \mathcal{F} \cap C_c(X)$ by (2.8) and the equality $\mathcal{F}_e \cap L^2(X, m) = \mathcal{F}$ from [24, Theorem 1.1.5-(iii)], and therefore for each open subset U of X we have

$$\{h \in (\mathcal{F}^{\mu})_e \mid h \text{ is } \mathcal{E}^{\mu}\text{-harmonic on } U\} = \{h \in \mathcal{F}_e \mid h \text{ is } \mathcal{E}\text{-harmonic on } U\}. \tag{4.9}$$

Note also that by [24, Theorem 5.2.11] we have the following (see [33, Section 2.1] and [24, Sections 1.2, 1.3 and 2.3] for the definition and basic properties of quasi-continuous functions):

(i) A subset N of X has 1-capacity zero with respect to $(X, d, m, \mathcal{E}, \mathcal{F})$, i.e., satisfies $\mathrm{Cap}_1(N) = 0$, if and only if N has 1-capacity zero with respect to $(X, \theta, \mu, \mathcal{E}^{\mu}, \mathcal{F}^{\mu})$. In other words, the notion of holding *q.e.*, i.e., holding outside a set of 1-capacity zero, with respect to $(X, d, m, \mathcal{E}, \mathcal{F})$ is equivalent to that with respect to $(X, \theta, \mu, \mathcal{E}^{\mu}, \mathcal{F}^{\mu})$.

(ii) A function $u : X \setminus N \to [-\infty, \infty]$ defined q.e. on X, where N is a subset of X with $\mathrm{Cap}_1(N) = 0$, is quasi-continuous with respect to $(X, d, m, \mathcal{E}, \mathcal{F})$ if and only if u is quasi-continuous with respect to $(X, \theta, \mu, \mathcal{E}^{\mu}, \mathcal{F}^{\mu})$.

Now choose a distortion function η so that $\mathrm{Id} : (X, \theta) \to (X, d)$ is an η-quasisymmetry, and suppose that $(X, d, m, \mathcal{E}, \mathcal{F})$ satisfies EHI with the constants $C > 1$ and $\delta \in (0, 1)$. Let $B_d(x, r)$ and $B_\theta(x, r)$ denote open balls of radius r centered at x in (X, d) and (X, θ), respectively. Let $x \in X$, $r > 0$, and let $h \in (\mathcal{F}^{\mu})_e = \mathcal{F}_e$ be $\mathcal{E}^{\mu}$-harmonic on $B_\theta(x, r)$ and satisfy $h \geq 0$ μ-a.e. on $B_\theta(x, r)$, where we consider only quasi-continuous m-versions of $h \in \mathcal{F}_e$ with respect to $(X, d, m, \mathcal{E}, \mathcal{F})$. Then h is $\mathcal{E}$-harmonic on $B_\theta(x, r)$ by (4.9), $h \geq 0$ q.e. on $B_\theta(x, r)$ by [33, Lemma 2.1.4] applied to $(X, \theta, \mu, \mathcal{E}^{\mu}, \mathcal{F}^{\mu})$, hence $h \geq 0$ m-a.e. on $B_\theta(x, r)$, and $B_\theta(x, \delta' r) \subset B_d(x, \delta t) \subset B_d(x, t) \subset B_\theta(x, r)$ for some $t > 0$ by (3.5) with $\delta' r$ in place of r and $A = \delta^{-1}$, where $\delta' := \eta^{-1}(\delta)$. Thus we obtain

$$m\text{-}\operatorname*{ess\,sup}_{B_\theta(x, \delta' r)} h \leq C \cdot m\text{-}\operatorname*{ess\,inf}_{B_\theta(x, \delta' r)} h \tag{4.10}$$

by EHI for $(X, d, m, \mathcal{E}, \mathcal{F})$ applied to $B_d(x, t)$ and h. On the other hand, we also have

$$m\text{-}\operatorname*{ess\,inf}_{B_\theta(x,\delta' r)} h \le h(y) \le m\text{-}\operatorname*{ess\,sup}_{B_\theta(x,\delta' r)} h,$$

first for m-a.e. $y \in B_\theta(x, \delta' r)$, then for q.e. $y \in B_\theta(x, \delta' r)$ by [33, Lemma 2.1.4] and hence for μ-a.e. $y \in B_\theta(x, \delta' r)$, and therefore from (4.10) we conclude that

$$\begin{aligned} h(y) &\le m\text{-}\operatorname*{ess\,sup}_{B_\theta(x,\delta' r)} h \\ &\le C \cdot m\text{-}\operatorname*{ess\,inf}_{B_\theta(x,\delta' r)} h \le Ch(z) \quad \text{for } \mu\text{-a.e. } y, z \in B_\theta(x, \delta' r), \end{aligned}$$

proving EHI for $(X, \theta, \mu, \mathcal{E}^\mu, \mathcal{F}^\mu)$. The converse implication from EHI for $(X, \theta, \mu, \mathcal{E}^\mu, \mathcal{F}^\mu)$ to EHI for $(X, d, m, \mathcal{E}, \mathcal{F})$ is proved in exactly the same way by noting that $\mathrm{Id} : (X, d) \to (X, \theta)$ is an $\tilde{\eta}$-quasisymmetry with the distortion function $\tilde{\eta}$ given by $\tilde{\eta}(t) := 1/\eta^{-1}(t^{-1})$. □

As mentioned in the introduction, Delmotte has constructed a space that satisfies EHI but fails to satisfy VD and hence fails to satisfy PHI(β) for any $\beta > 0$ [31]; see also [9, Example 8.4] for a similar construction. Nevertheless, it is possible to obtain PHI(β) after a time change and a change of the metric. We recall the characterization of EHI in [9,11].

Theorem 4.9 ([9,11]) *Let $(X, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space. Then the following are equivalent:*

(a) $(X, d, m, \mathcal{E}, \mathcal{F})$ satisfies the metric doubling property and EHI .
(b) There exist $\gamma > 2$, $\mu \in \mathcal{A}(X, d, m, \mathcal{E}, \mathcal{F})$ and $\theta \in \mathcal{J}(X, d)$ such that the time-changed MMD space $(X, \theta, \mu, \mathcal{E}^\mu, \mathcal{F}^\mu)$ satisfies PHI(γ)*. In other words, $d_{\mathrm{cw}} < \infty$.*

Moreover, either of these two conditions implies that (X, d) is arcwise connected and uniformly perfect.

Proof The implication from (a) to (b) follows from Remark 2.6, [9, Theorems 5.4 and 7.9] and Theorem 4.5. On the other hand, if (b) holds, then we see from Theorem 4.5 and Remark 3.18-(a) that (X, θ) is arcwise connected, uniformly perfect and doubling and that $(X, \theta, \mu, \mathcal{E}^\mu, \mathcal{F}^\mu)$ satisfies EHI, and therefore the same hold also for (X, d) and $(X, d, m, \mathcal{E}, \mathcal{F})$ by [40, Theorem 10.18] and Lemma 4.8, completing the proof. □

The following elementary lemma is used to verify that the function defined in (4.11) on a hyperbolic filling is gentle and satisfies the enhanced subadditive estimate (E).

Lemma 4.10 *Let (X, d, m) be a metric measure space that satisfies VD and let $\gamma > 0$. For any ball $B(x, r)$, we define*

$$\mathcal{C}(B(x, r)) = \frac{m(B(x, r))}{r^{\gamma}}. \tag{4.11}$$

(a) Let $\lambda \geq 1$. There exists $C_1 > 0$ (that depends only on the constant of VD and λ) such that for any $x, y \in X$ satisfying $B(x, \lambda r) \cap B(y, \lambda r) \neq \emptyset$, we have

$$\mathcal{C}(B(x, r)) \leq C_1 \mathcal{C}(B(y, r)).$$

(b) Let $a > 1$. There exists $C_2 \geq 1$ (that depends only on the constant of VD, γ and λ) such that for any $x, y \in X$ satisfying $y \in B(x, r)$, we have

$$C_2^{-1} \mathcal{C}(B(y, r/a)) \leq \mathcal{C}(B(x, r)) \leq C_2 \mathcal{C}(B(y, r/a)).$$

(c) There exists $C_3 > 1$ such that the following estimate holds: for all $a > 1$, $x \in X$, $r > 0$ and $z_1, \ldots, z_k$, $k \in \mathbb{N}$ such that $d(z_i, z_j) \geq r/(2a)$ for all $1 \leq i < j \leq k$ and satisfying $\cup_{i=1}^{k} B(z_i, r/a) \supset B(x, r/6)$, we have that

$$\mathcal{C}(B(x, r)) \leq C_3 a^{-\gamma} \sum_{i=1}^{k} \mathcal{C}(B(z_i, r/a)). \tag{4.12}$$

Proof We denote $m(B(x, r))$ by $V(x, r)$ in this proof.

(a) Let $C_D \in (1, \infty)$ denote the constant associated with VD and let $\alpha = \log_2 C_D$, so that by (3.39) we have

$$\frac{V(x, R)}{V(x, r)} \leq C_D \left(\frac{R}{r}\right)^{\alpha}, \quad \text{for all } 0 < r \leq R \text{ and } x \in X. \tag{4.13}$$

Let $z \in B(x, \lambda r) \cap B(y, \lambda r)$. By using $B(x, r) \subset B(z, (\lambda+1)r)$, $B(z, r) \subset B(y, (\lambda + 1)r)$ and (4.13), we obtain

$$V(x, r) \leq V(z, (\lambda + 1)r) \leq C_D(\lambda + 1)^{\alpha} V(z, r) \leq C_D^2 (\lambda + 1)^{2\alpha} V(y, r).$$

(b) Since $B(x, r) \subset B(y, 2r)$ and $B(y, r/a) \subset B(x, 2r)$, by (4.13) we have

$$V(x, r) \leq V(y, 2r) \leq C_D (2a)^{\alpha} V(y, r/a),$$
$$V(y, r/a) \leq V(x, 2r) \leq C_D V(x, r).$$

Therefore

$$\mathcal{C}(B(x,r)) \le C_D 2^{\alpha} a^{\alpha-\gamma} \mathcal{C}(B(y,r/a)), \quad \mathcal{C}(B(y,r/a)) \le C_D a^{\gamma} \mathcal{C}(B(x,r)).$$

(c) By VD and $\cup_{i=1}^{k} B(z_i, r/a) \supset B(x, r/3)$, we have

$$V(x,r) \le C_D^3 V(x, r/8) \le C_D^2 \sum_{i=1}^{k} V(x_i, r/a).$$

Dividing both sides by r^{γ}, we obtain (4.12) with $C_3 = C_D^3$.

□

The elliptic Harnack inequality implies that the capacities across annuli with similar locations and scales are comparable as we recall below.

Lemma 4.11 ([9, Lemmas 5.22 and 5.23]) *Let $(X, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space that satisfies the metric doubling property and EHI. Then for any $A_1, A_2 > 1$, there exist $C_1, C_2 > 1$ and $\gamma > 0$ such that for all $x, \in X$, and for any $0 < s \le r < \operatorname{diam}(X, d)/C_1$,, we have*

$$C_2^{-1} \left(\frac{r}{s}\right)^{-\gamma} \le \frac{\operatorname{Cap}(B(x,r), B(x, A_2 r)^c)}{\operatorname{Cap}(B(x,s), B(x, A_1 s)^c)} \le C_2 \left(\frac{r}{s}\right)^{\gamma}.$$

Proof This follows immediately from Remark 2.6 and [9, Theorem 5.4, Lemmas 5.22 and 5.23]. □

Using this lemma, we obtain the following comparison of capacity across annuli under a quasisymmetric change of metric.

Proposition 4.12 *Let $(X, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space that satisfies PHI(γ), where $\gamma \ge 2$. Let $\theta \in \mathcal{J}(X, d)$ and $a > 1$. Then there exists $C, A > 0$ such that the following property holds. For any $x, \widetilde{x} \in X$, $0 < r < \operatorname{diam}(X, \theta)/A$, $s > 0$, $n \in \mathbb{Z}$ such that*

$$s = \sup\{0 < t < 2\operatorname{diam}(X, d) : B_d(x,t) \subset B_\theta(x,r)\}, \tag{4.14}$$

and

$$2a^{-n-1} \le s < 2a^{-n}, d(\widetilde{x}, x) < 2a^{-n},$$

we have

$$C^{-1} \frac{m(B_d(\widetilde{x}, 2a^{-n}))}{\left[2a^{-n}\right]^{\gamma}} \le \operatorname{Cap}(B_\theta(x,r), B_\theta(x,2r)^c) \le C \frac{m(B_d(\widetilde{x}, 2a^{-n}))}{\left[2a^{-n}\right]^{\gamma}}. \tag{4.15}$$

Proof By Theorem 4.5, (X, d) satisfies doubling, uniformly perfect metric space. By Proposition 3.2, there exists $A_1, A_2, A_3 > 1$ such that for all $x \in X, 0 < r < \operatorname{diam}(X, \theta)$,

$$B_d(x, s) \subset B_\theta(x, r) \subset B_d(x, A_1 s) \subset B_d(s, 2A_1 s) \subset B_\theta(x, A_2 r) \subset B_d(x, A_3 s), \tag{4.16}$$

where $s > 0$ is as defined in (4.14). If $B_d(x, A_3 s) \neq X$ in (4.16), we have

$$\begin{aligned} \operatorname{Cap}(B_d(x, s), B_d(x, A_3 s)^c) &\leq \operatorname{Cap}(B_\theta(x, r), B_\theta(x, A_2 r)^c) \\ &\leq \operatorname{Cap}(B_d(x, A_1 s), B_d(x, 2A_1 s)^c). \end{aligned} \tag{4.17}$$

By Lemma 4.11, Proposition 3.2-(b), and Theorem 4.5, there exist $C_1, A > 1$ such that for all $x \in X, 0 < r < \operatorname{diam}(X, \theta)/A$, we have

$$C_1^{-1} \leq \frac{\operatorname{Cap}(B_\theta(x, r), B_\theta(x, 2r)^c)}{\operatorname{Cap}(B_d(x, s), B_d(x, A_3 s)^c)} \leq C_1, \tag{4.18}$$

and

$$C_1^{-1} \frac{m(B_d(x, s))}{s^\gamma} \leq \operatorname{Cap}(B_d(x, s), B_d(x, A_3 s)^c) \leq C_1 \frac{m(B_d(x, s))}{s^\gamma}, \tag{4.19}$$

where $s > 0$ is as given in (4.14). By (4.18), (4.19) and VD , we obtain (4.15). $\square$

We will use Theorem 3.24 to construct metrics. The following proposition plays a central role in constructing a function on the hyperbolic filling that satisfies the hypotheses (S1) and (S2) in Theorem 3.24.

Proposition 4.13 *Let $(X, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space that satisfies PHI(γ) for some $\gamma > 2$ and let $\lambda > 1$. There exist constants $A, C_1, C_2 > 1, \eta > 0$ (that depend only on λ and the constants associated with PHI(γ)) such that for any $a > 1, x \in X, 0 < r < \operatorname{diam}(X, d)/A$, and for any collection of balls $\mathcal{B} = \{B(y_i, r/a) : i \in I\}$ such that $\cup_{i \in I} B(y_i, r/a) = X$ and $\{B(y_i, r/(4a))\}$ is pairwise disjoint, there exists a function $\sigma : \mathcal{B} \to [0, \infty)$ that obeys the following properties (note that σ depends on $x \in X, r > 0$):*

(S1') *for any sequence of balls $\gamma = \{B_i : 1 \leq i \leq N\}$ in $\mathcal{B}$ such that $x_{B_1} \in B(x, r), x_{B_N} \notin B(x, 2r)$ and $\lambda \cdot B_i \cap \lambda \cdot B_{i+1} \neq \emptyset$ for all $i = 1, \ldots, N-1$, we have*

$$\sum_{i=1}^{N} \sigma(B_i) \geq 1, \tag{4.20}$$

and

$$\sigma(B) = 0, \quad \textit{for any ball } B \in \mathcal{B} \textit{ such that } x_B \notin B(x, 2r). \tag{4.21}$$

(S2') $\sigma : \mathcal{B} \to (0, \infty)$ *satisfies the following estimates*

$$\sum_{B \in \mathcal{B}} \sigma(B)^2 \frac{m(B)}{(r/a)^\gamma} \le C_1 \frac{m(B(x,r))}{r^\gamma} \tag{4.22}$$

and

$$\sup_{B \in \mathcal{B}} \sigma(B) \le C_2 a^{-\eta}. \tag{4.23}$$

In particular, for any $\beta > 2$, we have

$$\sum_{B \in \mathcal{B}} \sigma(B)^\beta \frac{m(B)}{(r/a)^\gamma} \le C_1 C_2^{\beta-2} a^{-(\beta-2)\eta} \frac{m(B(x,r))}{r^\gamma}. \tag{4.24}$$

Proof For a function $u \in \mathcal{F} \cap C(X)$ and $B \in \mathcal{B}$, we define its 'discretization' $u_d : \mathcal{B} \to \mathbb{R}$ as

$$u_d(B) := \fint_B u\, dm = \frac{1}{m(B)} \int_B u\, dm, \tag{4.25}$$

and its 'discrete gradient' $\sigma_u : \mathcal{B} \to [0, \infty)$

$$\sigma_u(B) := \sum_{B' \in \mathcal{B} : \lambda \cdot B' \cap \lambda \cdot B \neq \emptyset} \left| u_{\mathcal{B}}(B') - u_{\mathcal{B}}(B) \right|. \tag{4.26}$$

Our construction of σ is the discrete gradient σ_u of a well chosen function u. In particular, we choose a function $u \in \mathcal{F} \cap C_c(X)$ that satisfies the following properties: there exists $C_3 > 1, \eta > 0$ (that depends only on the constant associated with PHI(γ)) such that for all $x \in X$, $r < \operatorname{diam}(X, d)/A$, we have

$$u \equiv 1 \text{ on } B(x, 1.1r) \text{ and } u \equiv 0 \text{ on } B(x, 1.9r), \tag{4.27}$$

$$\mathcal{E}(u, u) \le C_3 \frac{m(B(x,r))}{r^\gamma}, \tag{4.28}$$

$$|u(y) - u(z)| \le C_3 \left(\frac{d(y,z)}{r} \right)^\eta \quad \text{for all } y, z \in X. \tag{4.29}$$

The existence of a function $u \in \mathcal{F} \cap C_c(X)$ satisfying the above properties follows from the cutoff Sobolev inequality CS(β), Theorem 4.5 and a standard covering argument as we recall below. By Theorem 4.5 we have that m is a doubling measure on (X, d) and hence (X, d) is a K_D-doubling metric space for some $K_D > 1$. Therefore there exists $N_D \in \mathbb{N}$ that depends only on K_D and $y_1, \ldots, y_{N_D} \in B(x, 1.1r)$ such that $\cup_{i=1}^{N_D} B(y_i, r/10) \supset B(x, 1.1r)$. By the construction of cutoff functions in [7, Section 3], there exists $C_4 > 0, \eta > 0$ such that for each $i = 1, \ldots, N_D$ satisfies

$$\phi_i \equiv 1 \text{ on } B(y_i, r/10), \quad \phi_i \equiv 0 \text{ on } B(y_i, r/5)^c,$$
$$\mathcal{E}(\phi_i, \phi_i) \le C_4 \frac{m(B(y_i, r/10))}{r^\gamma},$$
$$|\phi_i(y) - \phi_i(z)| \le C_4 \left(\frac{d(y, z)}{r}\right)^\eta \text{for all } y, z \in X.$$

By choosing $u = \max_{1 \le i \le N_D} \phi_i$ and using the above estimates along with triangle inequality, $\mathcal{E}(u, u) \le \sum_{i=1}^{N_D} \mathcal{E}(\phi_i, \phi_i)$, $|u(y) - u(z)| \le \max_{1 \le i \le N_D} |\phi_i(y) - \phi_i(z)|$, we obtain the desired properties (4.27), (4.28) and (4.29).

Let us show that the function $\sigma = \sigma_u$ as defined by (4.25) and (4.26) satisfies the desired conditions (S1') and (S2'). To this end, we note the following properties of $u_d \colon \mathcal{B} \to \mathbb{R}$:

$$\begin{aligned} u_d(B(x_B, r/a)) = 1 \quad & \text{for any } x_B \in B(x, r/10) \\ & (\text{since } B(x_B, r/a) \subset B(x, 1.1r)), \\ u_d(B(y, r/a)) = 0 \quad & \text{for any } y \in X \text{ such that } d(y, y') \le 2\lambda r/a, \\ & \text{where } y' \in B(x, 2r)^c (B(y, r/a) \subset B(x, 1.9r)^c \\ & \text{because } (2\lambda + 1)r/a < 0.1r \text{ by } (3.13)), \end{aligned}$$
$$\sum_{i=1}^{N} \sigma_u(B_i) \ge \sum_{i=1}^{N-1} |u_d(B_i) - u_d(B_{i+1})| \ge |u_d(B_1) - u_d(B_N)|$$

for any sequence of balls $B_1, \ldots, B_N \in \mathcal{B}$ such that $\lambda \cdot B_i \cap \lambda \cdot B_{i+1} \neq \emptyset$ for all $i = 1, \ldots, N - 1$. The above equations immediately imply (4.20) and (4.21).

Since the balls $B(y_i, r/(4a))$, $i \in I$ are disjoint, by doubling property of (X, d), there exists $C_5 > 1$ that depends only on λ and the doubling constant (but not on a) such that

$$\# \left\{ B' \in \mathcal{B} : \lambda \cdot B \cap \lambda \cdot B' \neq \emptyset \right\} \le C_5, \quad \text{for all } B \in \mathcal{B}. \tag{4.30}$$

For any two balls $B, B' \in \mathcal{B}$ such that $\lambda \cdot B \cap \lambda \cdot B' \neq \emptyset$, by (4.29) we have

$$\left| u_d(B) - u_d(B') \right| \le \sup_{y, z \le 2(\lambda+1)r/a} |u(y) - u(z)| \le C_3(2(\lambda + 1))^\eta a^{-\eta}. \tag{4.31}$$

Combining (4.26), (4.30) and (4.31), we obtain (4.23) for $\sigma = \sigma_u$.

It remains to show that $\sigma = \sigma_u$ satisfies (4.23). To this end, we recall the following Poincaré inequality PI(γ) implied by PHI(γ) and Theorem 4.5: there exist $C_P, A > 1$ such that

$$\begin{aligned}&\frac{1}{2m(B(y,s))}\int_{B(y,s)}\int_{B(y,s)}(f(z)-f(w))^2\,dm(z)\,dm(w)\\&=\int_{B(y,s)}(f-f_{B(y,s)})^2\,dm \le C_P s^\gamma \int_{B(y,As)} d\Gamma(f,f),\end{aligned}\tag{4.32}$$

for any $f \in \mathcal{F}$, $y \in X$, $s > 0$. On the basis of (4.32), the following comparison estimate between discrete and continuous energies is standard [6,28]. Similar to §3.2, for any two balls $B, B' \in \mathcal{B}$ by $B' \sim B$ we mean that $\lambda \cdot B \cap \lambda \cdot B' \neq \emptyset$. We obtain (4.22) by the following estimates:

$$\begin{aligned}&\sum_{B\in\mathcal{B}}\sigma_u^2(B)\frac{m(B)}{(r/a)^\gamma}\\
&\lesssim \sum_{B,B'\in\mathcal{B},B'\sim B}\left|u_d(B')-u_d(B)\right|^2\frac{m(B)}{(r/a)^\gamma}\\
&\quad\text{(by (4.30) and the Cauchy-Schwarz inequality)}\\
&\lesssim \sum_{B,B'\in\mathcal{B},B'\sim B}\frac{1}{(r/a)^\gamma m(B')}\int_B\int_{B'}(u(y)-u(z))^2\,dm(y)\,dm(z)\\
&\quad\text{(by Jensen's inequality)}\\
&\lesssim \sum_{B\in\mathcal{B}}\frac{1}{m((2\lambda+1)\cdot B)(r/a)^\gamma}\int_B\int_{(2\lambda+1)\cdot B}(u(y)-u(z))^2\,dm(y)\,dm(z)\\
&\quad\text{(by VD)}\\
&\lesssim \sum_{B\in\mathcal{B}}(2\lambda+1)^\gamma\int_{A(2\lambda+1)\cdot B}d\Gamma(u,u)\qquad\text{(by (4.32))}\\
&\lesssim \mathcal{E}(u,u)\\
&\quad\left(\text{since } (X,d) \text{ is } K_D\text{-doubling, we have } \sum_{B\in\mathcal{S}_n}\mathbb{1}_{(2A\lambda+A)\cdot B}\lesssim 1\right)\\
&\lesssim \frac{m(B(x,r))}{r^\gamma}\quad\text{(by (4.28))}.\end{aligned}$$

Finally, (4.24) follows from (4.22), (4.23), and

$$\sum_{B\in\mathcal{B}} \sigma(B)^{\beta} \frac{m(B)}{(r/a)^{\gamma}} \leq \left(\sup_{B\in\mathcal{B}} \sigma(B)\right)^{\beta-2} \sum_{B\in\mathcal{B}} \sigma(B)^{2} \frac{m(B)}{(r/a)^{\gamma}}.$$

□

The following proposition provides a convenient sufficient condition for a measure μ to be smooth and have full quasi-support.

Proposition 4.14 *Let $(X, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space that satisfies PHI(γ) for some $\gamma \geq 2$ and let $\theta \in \mathcal{J}(X, d)$. Let $\beta > 2$ and μ be a Borel measure on X that satisfies the following estimate: there exist $C_1, A > 1$ such that for any $x \in X$, $0 < r < \operatorname{diam}(X, \theta)/A$, we have*

$$C_1^{-1} \frac{\mu(B_\theta(x, r))}{r^\beta} \leq \operatorname{Cap}(B_\theta(x, r), B_\theta(x, 2r)^c) \leq C_1 \frac{\mu(B_\theta(x, r))}{r^\beta}.$$

Then $\mu \in \mathcal{A}(X, d, m, \mathcal{E}, \mathcal{F})$, i.e., μ is a smooth Radon measure on X with full quasi-support. Furthermore (X, θ, μ) is VD and RVD .

Proof By Theorem 4.5 and [11, Lemma 5.3] (see also Lemma 4.8 above), the MMD space $(X, \theta, m, \mathcal{E}, \mathcal{F})$ satisfies EHI and (X, d) is a doubling, uniformly perfect metric space, so that (X, θ) is also doubling and uniformly perfect by [40, Theorem 10.18 and Exercise 11.2]. The volume doubling property VD of μ in (X, θ) follows from Remark 2.6, Lemma 4.11 and [9, Lemma 6.3], and the RVD property of μ in (X, θ) follows from VD of (X, θ, μ), the uniform perfectness of (X, θ) and Remark 3.18-(b). That μ is a smooth Radon measure on X follows from Remark 2.6 and [9, Proposition 6.13], and μ has full quasi-support by Remark 2.6 and [9, Theorem 5.4 and Proposition 6.16]. □

4.2 Completion of the proof of Theorem 2.10

Now we are in the stage of completing the proof of our first main theorem (Theorem 2.10).

Proof of Theorem 2.10 By Theorem 4.9, it suffices to show that (a) implies (c). By Lemma 4.7, it suffices to show that $d_{\mathrm{cw}} \leq 2$. To this end, we fix an arbitrary $\beta > 2$. We shall construct a metric $\theta \in \mathcal{J}(X, d)$ and a measure $\mu \in \mathcal{A}(X, d, m, \mathcal{E}, \mathcal{F})$ such that the time-changed MMD space $(X, \theta, \mu, \mathcal{E}^\mu, \mathcal{F}^\mu)$ satisfies PHI(β).

By Theorem 4.9, (1.3), (2.9), and by changing the metric and measure if necessary, we may assume that $(X, d, m, \mathcal{E}, \mathcal{F})$ satisfies PHI(γ) for some $\gamma > 2$. By Theorem 4.5, $(X, d, m, \mathcal{E}, \mathcal{F})$ satisfies VD, RVD, EHI and cap(γ).

If (X, d) is bounded, we scale the metric so that $\mathrm{diam}(X, d) = \frac{1}{2}$. By completeness and the metric doubling property, we recall that (X, d) is compact [40, Definition 10.15 and Exercise 10.17].

Fix $\lambda \geq 32$ and let a be an arbitrary constant that satisfies (3.13). The choice of a will be made later in the proof. Let $x_0 \in X$. Let $\mathcal{S} = \coprod_{k\in\mathbb{Z}_{\geq 0}} \mathcal{S}_k$ denote the vertex set of the hyperbolic filling as defined in Definition 3.10, where $\mathcal{S}_k = \left\{B(x, 2a^{-k}) : x \in N_k\right\}$, where $N_k, k \in \mathbb{Z}$ is a sequence of a^{-k}-separated sets such that $N_k \subset N_{k+1}$ and $x_0 \in N_k$ for all $k \in \mathbb{Z}$ (recall from Lemma 3.9 this is a hyperbolic filling in the sense of Definition 3.6).

We define a function $\mathcal{C} : \coprod_{k\in\mathbb{Z}} \mathcal{S}_k \to (0, \infty)$ on the extended hyperbolic filling by

$$\mathcal{C}(B(x, 2a^{-k})) = \frac{m(B(x, 2a^{-k}))}{(2a^{-k})^{\gamma}}, \quad \text{for any } k \in \mathbb{Z} \text{ and for any } B(x, 2a^{-k}) \in \mathcal{S}_k. \tag{4.33}$$

Let us verify that $\mathcal{C}$ is gentle and satisfies the enhanced subadditivity property (E). By Lemma 4.10-(a),(b), there exist K_h, K_v such that K_v depends only on a and the constant associated with VD, K_h depends only on λ and the constant associated with VD such that

$$\begin{aligned} \mathcal{C}(B_1) &\leq K_h \mathcal{C}(B_2), \quad \text{whenever } B_1 \text{ and } B_2 \text{ share a horizontal edge,} \\ \mathcal{C}(B_1) &\leq K_v \mathcal{C}(B_2), \quad \text{whenever } B_1 \text{ and } B_2 \text{ share a vertical edge.} \end{aligned} \tag{4.34}$$

Recall that for every ball $B \in \mathcal{S}_k$, $k \in \mathbb{Z}$, there exists an unique ball $g(B)_{k-1} \in \mathcal{S}_{k-1}$ such that there is a vertical edge between $g(B)_{k-1}$. Note that is $B \sim B'$ and $B, B' \in \mathcal{S}_{k+1}$, then by (3.13), we have $d(x_B, x_{B'}) \leq 2\lambda a^{-k-1} \leq \frac{1}{12} a^{-k}$. We denote the set of all non-peripheral elements of $\coprod_{k\in\mathbb{Z}} \mathcal{S}_k$ by $\mathcal{N}$ as given in Definition 3.23. Hence if $C = g(B)_k$ and $d(x_C, x_B) < \frac{1}{6}(2a^{-k}) + 2a^{-k-1}$, then $d(x_C, x_{B'}) < (3^{-1} + 2a^{-1} + 12^{-1})a^{-k} < \frac{1}{2}a^{-k}$, and hence $B \in \mathcal{N}$. This along with Lemma 4.10-(c) imply that, there exists $C_1 > 0$ such that

$$\mathcal{C}(B) \leq C_1 a^{-\gamma} \sum_{B' \in \mathcal{N} \cap \mathcal{D}_{k+1}(B)} \mathcal{C}(B'), \quad \text{for all } B \in \mathcal{S}_k, k \in \mathbb{Z}. \tag{4.35}$$

By (4.34) and (4.35), we conclude that $\mathcal{C}$ is gentle and satisfies the enhanced subadditivity property (E).

For $k \in \mathbb{Z}$ and $B \in \mathcal{S}_k$, let $\Gamma_{k+1}(B)$ denote the set of horizontal paths $\gamma = \{B_i\}_{i=1}^N$, $N \geq 2$ as given in Definition 3.22. If $\mathrm{diam}(X, d) = \frac{1}{2}$, we note that $\Gamma_k(B) = \emptyset$ for all $k \le 0$. If $\mathrm{diam}(X, d) = \frac{1}{2}$, we assume that

$$a > 2A, \tag{4.36}$$

where A is the constant in Proposition 4.13. If either $k \in \mathbb{Z}$, $\mathrm{diam}(X, d) = \infty$ or if $k \in \mathbb{N}$, $\mathrm{diam}(X, d) = \frac{1}{2}$, for any $B \in \mathcal{S}_k$, we define $\sigma_B : \mathcal{S}_{k+1} \to (0, \infty)$ as the function defined in Proposition 4.13, that satisfies

$$\sum_{i=1}^{N} \sigma_B(B_i) \ge 1, \quad \text{for any } \{B_i\}_{i=1}^{N} \in \Gamma_{k+1}(B).$$

Otherwise if $\mathrm{diam}(X, d) = \frac{1}{2}$ and $k \ge 0$, we simply define $\sigma_B : \mathcal{S}_{k+1} \to [0, \infty)$ as $\sigma_B \equiv 0$ for all $B \in \mathcal{S}_k$. For any $k \in \mathbb{Z}$ and for any $B \in \mathcal{S}_k$, we define

$$\sigma(B) = \max_{C \in \mathcal{S}_{k-1}} \sigma_C(B), \quad \text{for any } k \in \mathbb{Z},\ B \in \mathcal{S}_k. \tag{4.37}$$

Evidently, by Proposition 4.13, we have

$$\sum_{i=1}^{N} \sigma(B_i) \ge 1, \quad \text{for any } \{B_i\}_{i=1}^{N} \in \Gamma_{k+1}(B) \text{ and for any } k \in \mathbb{Z},\ B \in \mathcal{S}_k. \tag{4.38}$$

In the compact case, the above statement is vacuously true for $k \le 0$. By Proposition 4.13 and the argument in Lemma 3.29, there exist $C_2, \eta > 0$ such that

$$\sum_{B' \in \mathcal{D}_{k+1}(B)} \sigma(B')^{\beta} \mathcal{C}(B') \le C_2 a^{-(\beta-2)\eta} \mathcal{C}(B), \quad \text{for any } k \in \mathbb{Z},\ B \in \mathcal{S}_k, \tag{4.39}$$

where $\mathcal{D}_{k+1}(B)$ denote the set of descendants of B in $\mathcal{S}_{k+1}$ (that is, $\mathcal{D}_{k+1}(B)$ is the set of elements in $\mathcal{S}_{k+1}(B)$ that share a vertical edge with B).

We consider two cases.

Case 1: (X, d) is bounded. Let $\mathcal{S} = \coprod_{k \ge 0} \mathcal{S}_k$ denote the vertex set of the hyperbolic filling. In this case by (4.35), we can ensure the enhanced sub-additivity estimate (E) by choosing a large enough. Similarly by (4.38) and (4.39), the function σ defined above satisfies the hypotheses (S1) and (S2) of Theorem 3.24 for all large enough a. Therefore by Theorems 3.24 and 3.14, and Proposition 3.21, there exist a metric $\theta \in \mathcal{J}(X, d)$ and a measure μ on X

that satisfies the following estimate: there exists $C_3 > 0$ such that

$$C_3^{-1} r^\beta \mathcal{C}(B) \le \mu(B_\theta(x,r)) \le C_3 r^\beta \mathcal{C}(B) \tag{4.40}$$

for all $x \in X, r < \operatorname{diam}(X,\theta)$, $B \in \mathcal{A}_{\mathcal{S}}(B_d(x,s))$, where s is the largest number in $(0, 2\operatorname{diam}(X,d)]$ such that $B_d(x,s) \subset B_\theta(x,r)$ (as defined in (3.4)) and $\mathcal{A}_{\mathcal{S}}(B_d(x,s))$ is as given in Definition 3.16. Combining (4.40) and Proposition 4.12, we see that there exist $A_1, C_4 > 0$ such that

$$C_4^{-1} \frac{\mu(B_\theta(x,r))}{r^\beta} \le \operatorname{Cap}(B_\theta(x,r), B_\theta(x,2r)^c) \le C_4 \frac{\mu(B_\theta(x,r))}{r^\beta}, \tag{4.41}$$

for any $x \in X, 0 < r < \operatorname{diam}(X,d)/A_1$. By (4.41) and Proposition 4.14, $\mu \in \mathcal{A}(X,d,m,\mathcal{E},\mathcal{F})$, $(X,\theta,\mu,\mathcal{E}^\mu,\mathcal{F}^\mu)$ satisfies VD, RVD and cap(β), and by Lemma 4.8 it also satisfies EHI.

Thus by Theorem 4.5 the MMD space $(X,\theta,\mu,\mathcal{E}^\mu,\mathcal{F}^\mu)$ satisfies PHI(β). Since $\beta > 2$ is arbitrary, we conclude that the conformal walk dimension is two.

Case 2: (X,d) is unbounded.

The approach in the unbounded case is to construct metrics and measures on an increasing sequence of compact sets that cover X, and to take suitable sub-sequential limit. Let $x_0 \in X$ be the point such that $x_0 \in N_k$ for all $k \in \mathbb{Z}$ as given in Definition 3.10. We consider the sequence of subsets

$$X_n = \overline{D_{-n}(x_0)}, \quad \text{for any } n \in \mathbb{N}, \tag{4.42}$$

where $D_{-n}(x_0)$ is as defined in (3.17). By Lemma 3.9-(c), X_n is compact and satisfies

$$B(x_0, (2^{-1} - (a-1)^{-1})a^n) \subset X_n \subset \overline{B}(x_0, (1-a^{-1})^{-1}a^n). \tag{4.43}$$

For any $k \ge -n$, we define

$$\mathcal{S}_k^{(n)} = \left\{ B(x, 2a^{-k}) \cap X_n : x \in N_k \cap D_{-n}(x_0) \right\}.$$

By Lemma 3.9, $\mathcal{S}^{(n)} := \coprod_{k\in\mathbb{Z}, k\ge -n} \mathcal{S}_k^{(n)}$ is a hyperbolic filling of the compact space with the same vertical edges induced from the extended hyperbolic filling. Similarly, $B \cap X_n, B' \cap X_n \in \mathcal{S}_k^{(n)}$ share a horizontal edge if and only if $(\lambda \cdot B \cap X_n) \cap (\lambda \cdot B' \cap X_n) \neq \emptyset$. We define $\mathcal{C}_n : \mathcal{S}^{(n)} \to (0,\infty)$, $\sigma_n : \mathcal{S}^{(n)} \to [0,\infty)$ as

$$\mathcal{C}_n(B \cap X_n) = \mathcal{C}(B), \quad \sigma_n(B \cap X_n) = \sigma(B),$$

for any $B \in B \in \coprod_{k \geq -n} \mathcal{S}_k$, where $\mathcal{C}$ is as given in (4.33) and σ is a s given in (4.37). Similar to the compact case, by choosing $a > 1$ large enough, by (4.35), we obtain the enhanced subadditivity estimate (E) for $\mathcal{C}_n$ uniformly over n (that is, the constant δ in associated with (E) does not depend on n) Similarly, by increasing a is necessary, and by (4.38) and (4.39), the function σ_n defined above satisfies the hypotheses (S1) and (S2) of Theorem 3.24 uniformly in n (that is, the constant η_0 in associated with (S2) does not depend on n).

Similar to the compact case, by Theorems 3.24 and 3.14, there exist metrics $\theta_n \in \mathcal{J}(X_n, d)$ for each $n \in \mathbb{N}$, and a distortion function $\eta : [0, \infty) \to [0, \infty)$ such that the identity map

$$\mathrm{Id} : (X_n, d) \to (X_n, \theta_n) \text{ is an } \eta\text{-quasisymmetry for each } n \in \mathbb{N}. \tag{4.44}$$

By Proposition 3.21, there exist measures μ_n on X_n for each $n \in \mathbb{N}$, constant $C_5 > 0$ such that

$$C_5^{-1} r^\beta \mathcal{C}_n(B) \leq \mu_n(B_{\theta_n}(x, r)) \leq C_5 r^\beta \mathcal{C}_n(B), \tag{4.45}$$

for all $x \in X_n$, $r < \mathrm{diam}(X_n, \theta)$, $B \in \mathcal{A}_{\mathcal{S}}(B_d(x, s))$, where s is the largest number in $[0, 2\,\mathrm{diam}(X_n, d)]$ such that $B_d(x, s) \cap X_n \subset B_\theta(x, r)$ (as defined in (3.4)) and $\mathcal{A}_{\mathcal{S}}(B_d(x, s) \cap X_n)$ is as given in Definition 3.16 corresponding to the hyperbolic filling $\mathcal{S}^{(n)}$ of X_n.

Next, normalize the metrics and measures by choosing a pair of sequences $\beta_n, \gamma_n > 0$ such that $\widehat{\theta}_n = \beta_n \theta_n$, $\widehat{\mu}_n = \gamma_n \mu_n$ satisfy

$$\mathrm{diam}(B_d(x_0, 1), \widehat{\theta}_n) = 1, \quad \widehat{\mu}_n(B_d(o, 1)) = 1. \tag{4.46}$$

By (4.43) and (3.13), we note that $B(x_0, 1) \subset X_n$ for all $n \in \mathbb{N}$.

We choose $p \in X$ such that $d(x_0, p) = \frac{1}{2}$. Since $\mathrm{Id} : (X_n, d) \to (X_n, \widehat{\theta}_n)$ is a η-quasisymmetry, by comparing the ratio of the diameter of the sets $\{x_0, p\} \subset X_1$ in the metrics d and θ_n using (3.7), there exists $C_{x_0, p} > 1$ such that

$$C_{x_0,p}^{-1} \leq \widehat{\theta}_n(o, p) \leq C_{x_0,p}^{-1}, \text{ for all } n \in \mathbb{N}.$$

We estimate $\widehat{\theta}_n(x, y)/\widehat{\theta}_n(x_0, p)$ by writing it as $\frac{\theta_n(x,y)}{\widehat{\theta}(x_0,x)} \frac{\theta_n(x_0,x)}{\widehat{\theta}(x_0,p)}$ and using η-quasisymmetry to estimate each of the factors and their reciprocals. This yields the following estimate: for any $x, y \in X_n$, $n \in \mathbb{N}$, there exists $C_{x,y} > 1$ ($C_{x,y}$ depends only on $d(x_0, x)$, $d(x, y)$ and η) such that

$$C_{x,y}^{-1} \leq \widehat{\theta}_n(x, y) \leq C_{x,y}^{-1}, \text{ for all } n \in \mathbb{N} \text{ such that } \{x, y\} \subset X_n. \tag{4.47}$$

By a similar computation, for any $(x, y) \in X \times X$ and for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $(x', y') \in X \times X$ with $d(x, x') \vee d(y, y') < \delta$, we have

$$\left|\widehat{\theta}_n(x,y)-\widehat{\theta}_n(x',y')\right| \le \widehat{\theta}_n(x,x')+\widehat{\theta}_n(y,y') < \varepsilon, \tag{4.48}$$

for all $n \in \mathbb{N}$ such that $x, x', y, y' \in X_n$. By (4.48), (4.47) and Arzelà-Ascoli theorem on the product space $X_n \times X_n$ equipped with the product metric $d_\infty((x,y),(x',y')) = d(x,x') \vee d(y,y')$, the sequence of functions $\widehat{\theta}_m, m \ge n$, has a subsequence that converges uniformly to a metric θ in X_n. By a diagonalization argument, we obtain a subsequence of $\left\{\widehat{\theta}_n : n \in \mathbb{N}\right\}$, that converges uniformly in compact subsets of $X \times X$. The limit metric $\theta \in \mathcal{J}(X,d)$ and $\mathrm{Id} : (X,d) \to (X,\theta)$ is an η-quasisymmetry.

The measures $\widehat{\mu}_n$ constructed using Lemma 3.20 are uniformly doubling in the following sense: there exists $C_D \ge 1$ such that for all $n \in \mathbb{N}$, $x \in X_n$, $0 < r < \operatorname{diam}(X_n, \widehat{\theta}_n)$, we have

$$\frac{\widehat{\mu}_n\left(B_{\widehat{\theta}_n}(x,2r)\right)}{\widehat{\mu}_n\left(B_{\widehat{\theta}_n}(x,r)\right)} \le C_D.$$

By the argument in [74, Theorem 1], by a further diagonalization argument using weak*-compactness of $\{\widehat{\mu}_m : m \ge n\}$ on X_n for all $n \in \mathbb{N}$, we obtain a measure μ on X. By (4.45) and Proposition 4.12, there exist constants $C_6 > 1, A_2 > 1$ such that

$$C_6^{-1}\frac{\mu(B_\theta(x,r))}{r^\beta} \le \operatorname{Cap}(B_\theta(x,r), B_\theta(x,2r)^c) \le C_4\frac{\mu(B_\theta(x,r))}{r^\beta}, \tag{4.49}$$

for any $x \in X, r > 0$. The remainder of the proof is exactly same as the compact case. Hence, we conclude that the conformal walk dimension is two. □

5 The attainment and Gaussian uniformization problems

In this section, we introduce the attainment problem for the conformal walk dimension and the Gaussian uniformization problem. Then we discuss partial progress towards them.

Let $(X, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space that satisfies the metric doubling property and EHI. Recall that the Gaussian uniformization problem ask for a description of all metrics $\theta \in \mathcal{J}(X,d)$ and measures $\mu \in \mathcal{A}(X,d,m,\mathcal{E},\mathcal{F})$ such that the corresponding time-changed MMD space $(X, \theta, \mu, \mathcal{E}^\mu, \mathcal{F}^\mu)$ satisfies PHI(2). For any $\beta > 0$, we define

$$\mathcal{G}_\beta(X,d,m,\mathcal{E},\mathcal{F}) := \left\{ \mu \;\middle|\; \begin{array}{l} \mu \in \mathcal{A}(X,d,m,\mathcal{E},\mathcal{F}), \text{ there exists } \theta \in \mathcal{J}(X,d) \\ \text{such that } (X,\theta,\mu,\mathcal{E}^\mu,\mathcal{F}^\mu) \text{ satisfies PHI}(\beta) \end{array} \right\}. \tag{5.1}$$

We define the set of *Gaussian admissible measures* as

$$\mathcal{G}(X,d,m,\mathcal{E},\mathcal{F}) := \mathcal{G}_2(X,d,m,\mathcal{E},\mathcal{F}). \tag{5.2}$$

By Theorem 2.10, we have

$$\begin{aligned} &\mathcal{G}_\beta(X,d,m,\mathcal{E},\mathcal{F}) = \emptyset \text{ for any } \beta < 2, \text{ and} \\ &\mathcal{G}_\beta(X,d,m,\mathcal{E},\mathcal{F}) \neq \emptyset \text{ for any } \beta > 2. \end{aligned}$$

This raises the following questions:

1. **Attainment problem**: Is $\mathcal{G}(X,d,m,\mathcal{E},\mathcal{F}) \neq \emptyset$? Or equivalently, is the infimum in (1.4) attained?
2. **Gaussian uniformization problem**: Describe all measures in the set $\mathcal{G}(X,d,m,\mathcal{E},\mathcal{F})$.

By Proposition 2.11-(a), the Gaussian admissible measures can be described as

$$\mathcal{G}(X,d,m,\mathcal{E},\mathcal{F}) = \left\{ \mu \;\middle|\; \begin{array}{l} \mu \in \mathcal{A}(X,d,m,\mathcal{E},\mathcal{F}),\, d_{\mathrm{int}}^\mu \in \mathcal{J}(X,d), \\ (X,d_{\mathrm{int}}^\mu,\mu,\mathcal{E}^\mu,\mathcal{F}^\mu) \text{ satisfies PHI}(2) \end{array} \right\}, \tag{5.3}$$

where d_{int}^μ denotes the intrinsic metric of the MMD space $(X,d,\mu,\mathcal{E}^\mu,\mathcal{F}^\mu)$ (recall Definition 2.3).

In this section, we prove Theorem 2.12 and discuss its consequences for the Gaussian uniformization problem. In particular, Theorem 2.12 shows that any two measures $\mu_1, \mu_2 \in \mathcal{G}(X,d,m,\mathcal{E},\mathcal{F})$ must be A_∞-related in (X,d) (provided such measures exist).

5.1 Consequences of PHI(2)

We begin with the proof of Proposition 2.11, which is essentially contained in [56, Section 4]. This states that any measure on $\mathcal{G}(X,d,m,\mathcal{E},\mathcal{F})$ must be a minimal energy-dominant measure and that the metric must be bi-Lipschitz equivalent to the intrinsic metric of the time-changed MMD space.

Proof of Proposition 2.11 By PHI(2) and Theorem 4.5, we have VD and all of the equivalent conditions in [37, Theorem 1.2].

(a) We use [80, Theorem 1.6 and Remark 1.7-(a)] and [56, Proposition 4.8] to obtain (a).
(b) This follows from [56, Propositions 4.5 and 4.7].

□

Definition 5.1 Let (X, d) be a metric space and let $u\colon X \to \mathbb{R}$. We define the *pointwise Lipschitz constant* $\operatorname{Lip} u(x)$ of u at $x \in X$ as

$$\operatorname{Lip} u(x) := \limsup_{y\to x} \frac{|u(x) - u(y)|}{d(x, y)},$$

and $\operatorname{Lip}(X)$ denotes the collection of all functions $u\colon X \to \mathbb{R}$ with

$$\|u\|_{\operatorname{Lip}(X)} := \sup_{x,y\in X, x\neq y} \frac{|u(x) - u(y)|}{d(x, y)} < \infty.$$

When it is necessary, we also write Lip as Lip_d to specify the metric d.

We recall the notion of upper gradient and its variants. We refer the reader to [40,43] for a comprehensive account.

Definition 5.2 Let (X, d, m) be a metric measure space and let $u\colon X \to \mathbb{R}$ be a Borel measurable function. A non-negative Borel measurable function g is called an *upper gradient*[4] if

$$|u(x) - u(y)| \leq \int_\gamma g\, ds,$$

for every rectifiable curve γ between x and y. A non-negative Borel measurable function g is called a *p-weak upper gradient of u* with $p \in [1, \infty)$ if

$$|u(x) - u(y)| \leq \int_\gamma g\, ds,$$

for all $\gamma \in \Gamma_{\mathrm{rect}} \setminus \Gamma_0$, where x and y are the endpoints of γ, Γ_{rect} denotes the collection of non-constant compact rectifiable curves and Γ_0 has p-modulus zero in the sense that

$$\inf\left\{ \|\rho\|_{L^p(X)}^p \;\middle|\; \begin{array}{l} \rho\colon X \to [0, \infty],\ \rho \text{ is Borel measurable,} \\ \int_\gamma \rho\, ds \geq 1 \text{ for all } \gamma \in \Gamma_0 \end{array} \right\} = 0.$$

We denote by $N^{1,p}(X)$ the collection of functions $u \in L^p(X)$ that have a p-weak upper gradient $g \in L^p(X)$, and define $\|u\|_{N^{1,p}(X)} = \|u\|_{L^p(X)} +$

4 This notion is called *very weak gradient* in [42]. Our terminology is borrowed from [43].

$\inf_g \|g\|_{L^p(X)}$, where g is taken over all p-weak upper gradients of u. We denote by $N^{1,p}_{\mathrm{loc}}(X)$ the class of functions $u \in L^p_{\mathrm{loc}}(X)$ that have a p-weak upper gradient that belongs to $L^p(B)$ for each ball B. If necessary, we denote the spaces $N^{1,p}(X)$ and $N^{1,p}_{\mathrm{loc}}(X)$ by $N^{1,p}(X, d, m)$ and $N^{1,p}_{\mathrm{loc}}(X, d, m)$ respectively.

Definition 5.3 We say that (X, d, m) supports a $(1, p)$-*Poincaré inequality* with $p \in [1, \infty)$ if there exists constants $K \geq 1, C > 0$ such that for all $u \in \mathrm{Lip}(X)$, $x \in X$ and $r > 0$,

$$\fint_{B(x,r)} \left|u - u_{B(x,r)}\right| dm \leq Cr \left[\fint_{B(x,Kr)} (\mathrm{Lip}(u))^p \; dm\right]^{1/p},$$

where $\fint_A f \, dm$ denotes $\frac{1}{m(A)} \int_A f \, dm$ and $u_{B(x,r)} = \fint_{B(x,r)} u \, dm$. It is known that (X, d, m) supports a $(1, p)$-Poincaré inequality if and only if there exists constants $K \geq 1, C > 0$ such that for every function u that is integrable on balls and for any upper gradient g of u in X, $x \in X$ and $r > 0$,

$$\fint_{B(x,r)} \left|u - u_{B(x,r)}\right| dm \leq Cr \left[\fint_{B(x,Kr)} g^p \, dm\right]^{1/p},$$

where $u_{B(x,r)}$ is as above [43, Theorem 8.4.2].

We need the following self-improvement of Poincaré inequality.

Proposition 5.4 *Let* $(X, d, m, \mathcal{E}, \mathcal{F})$ *be an MMD space that satisfies* PHI(2). *Then we have the following:*

(a) (Cf. [69, Theorem 2.2]) $\mathcal{F} = N^{1,2}(X)$ *with equivalent norms,* $\mathrm{Lip}(X) \cap C_c(X)$ *is dense in* $\mathcal{F}$ *and* $\mathcal{F}_{\mathrm{loc}} = N^{1,2}_{\mathrm{loc}}(X)$.
(b) (Cf. [59, Theorem 1.0.1]) (X, d, m) *satisfies* $(1, p)$-*Poincaré inequality for some* $p \in [1, 2)$.

Proof (a) Let d_{int} denote the intrinsic metric corresponding to the MMD space PHI(2). Since VD and PI(2) are preserved under a bi-Lipschitz change of the metric (cf. [43, Lemma 8.3.18]), by Proposition 2.11-(a), the MMD space $(X, d_{\mathrm{int}}, m, \mathcal{E}, \mathcal{F})$ also satisfies VD and PI(2). Therefore by [69, Theorem 2.2], $\mathcal{F} = N^{1,2}(X, d_{\mathrm{int}}, m)$ with equivalent norms and $\mathrm{Lip}_{d_{\mathrm{int}}}(X) \cap C_c(X)$ is dense in $\mathcal{F}$ and $\mathcal{F}_{\mathrm{loc}} = N^{1,2}_{\mathrm{loc}}(X, d_{\mathrm{int}}, m)$. Since d and d_{int} are bi-Lipschitz equivalent, $\mathrm{Lip}_{d_{\mathrm{int}}}(X) = \mathrm{Lip}_d(X)$ and $\mathrm{Lip}_d(u)$ is comparable to $\mathrm{Lip}_{d_{\mathrm{int}}}(u)$; that is there exists $C > 0$ such that

$$C^{-1}\,\mathrm{Lip}_d(u)(x) \le \mathrm{Lip}_{d_{\mathrm{int}}}(u)(x) \le C\,\mathrm{Lip}_d(u)(x) \quad \text{for all } x \in X,\, u \in \mathrm{Lip}_d(X). \tag{5.4}$$

(b) By (a), [59, Proposition 2.1] and [43, Lemma 8.3.18], (X, d, m) satisfies the $(1, 2)$-Poincaré inequality. By the self-improving property of [59, Theorem 1.0.1], (X, d, m) satisfies $(1, p)$-Poincaré inequality for some $p \in [1, 2)$. $\square$

5.2 A_∞-weights and the Gaussian uniformization problem

Definition 5.5 (A_∞-relation) Let (X, d, m) be a complete metric measure space such that m is a doubling measure. Let m' be another doubling Borel measure on X. Then m' is said to be *A_∞-related* to m if for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$m(E) < \delta m(B) \quad \text{implies} \quad m'(E) < \varepsilon m'(B)$$

whenever E is a measurable subset of a ball B. Evidently, if m' is A_∞-related to m, then m' is absolutely continuous with respect to m, so that $dm' = w\, dm$ for some nonnegative locally integrable weight function w. It turns out that being A_∞-related is a symmetric relation among doubling measures; that is, if m' is A_∞-related to m, then m is A_∞-related to m' [86, Chapter I].

Consider the following *reverse Hölder inequality*: there is a locally m-integrable function w in X together with constants $C \ge 1$ and $p > 1$ such that $dm' = w\, dm$ and

$$\left(\fint_B w^p\, dm \right)^{1/p} \le C \fint_B w\, dm \tag{5.5}$$

whenever B is a ball in (X, d). It is well known that a doubling measure m' is A_∞-related to m if and only if m' is non-zero and the reverse Hölder inequality (5.5) is satisfied [86, Chapter I].

The A_∞-relation among doubling measures is preserved under quasisymmetric change of metric as we show below.

Lemma 5.6 *Let d_1, d_2 two quasisymmetric metrics on X such that the metrics d_1, d_2 are uniformly perfect. Let m_1, m_2 be two doubling Borel measures with respect to d_1 such that m_1 and m_2 are A_∞-related with respect to the metric d_1. Then m_1 and m_2 are A_∞-related with respect to d_2.*

Proof Let $B_i(x, r)$ denote the open ball in metric d_i for $i = 1, 2$. By [75, Lemma 1.2.18], there exists $C > 0$ such that the following holds: for each $x \in X, r > 0$ and $i \in \{1, 2\}$, there exists $s > 0$ such that

$$B_{3-i}(x, C^{-1}s) \subseteq B_i(x,r) \subseteq B_{3-i}(x, Cs), \quad \text{for all } x \in X, r > 0. \tag{5.6}$$

Note that since m_1 and m_2 are doubling with respect to d_1, they are also doubling on (X, d_2). Therefore, there exists $C_1 > 0$ such that

$$\frac{m_i(B_j(x, Cr))}{m_i(B_j(x, r))} \le C_1, \quad \text{for all } x \in X, r > 0, \text{and } i, j \in \{1, 2\}. \tag{5.7}$$

Since m_1 and m_2 are A_∞-related in (X, d_1), we have $m_2 \ll m_1, dm_2 = w\, dm_1$, where $w \ge 0$ is a Borel measurable function that satisfies the following reverse Hölder inequality: there exists $C_R \ge 1$, $p > 1$ such that

$$\left(\fint_{B_1(x,r)} w^p \, dm_1 \right)^{1/p} \le C_R \fint_{B_1(x,r)} w \, dm_1, \quad \text{for all } x \in X, r > 0. \tag{5.8}$$

For all $x \in X, r > 0$, we estimate

$$\begin{aligned}
\left(\fint_{B_2(x,r)} w^p \, dm_1 \right)^{1/p} &\le \left(\fint_{B_1(x,Cs)} w^p \, dm_1 \right)^{1/p} \left(\frac{m_1(B_1(x, Cs))}{m_1(B_1(x, C^{-1}s))} \right)^{1/p} \\
&\quad \text{(by (5.6))} \\
&\le C_R C_1^{2/p} \fint_{B_1(x,Cs)} w \, dm_1 \quad \text{(by (5.8) and (5.7))} \\
&= C_R C_1^{2/p} \frac{m_2(B_1(x, Cs))}{m_1(B_1(x, Cs))} \quad (\text{by } dm_2 = w \, dm_1) \\
&\le C_R C_1^{2/p} \frac{m_2(B_1(x, Cs))}{m_1(B_2(x, r))} \quad \text{(by (5.6))} \\
&\le C_R C_1^{4/p} \frac{m_2(B_1(x, C^{-1}s))}{m_1(B_2(x, r))} \quad \text{(by (5.7))} \\
&\le C_R C_1^{4/p} \frac{m_2(B_2(x, r))}{m_1(B_2(x, r))} \quad \text{(by (5.6))} \\
&\le C_R C_1^{4/p} \fint_{B_2(x,r)} w \, dm_1, \quad (\text{since } dm_2 = w \, dm_1).
\end{aligned}$$

□

Let $f : (X_1, d_1) \to (X_2, d_2)$ be a homeomorphism between two metric spaces. For all $x \in X, r > 0$, we define

$$L_f(x,r) = \sup \{ d_2(f(x), f(y)) : d_1(x, y) \le r \}, \; L_f(x) = \limsup_{r \to 0} \frac{L_f(x,r)}{r}. \tag{5.9}$$

For $\varepsilon > 0$, we define

$$L_f^\varepsilon(x) = \sup_{0<r\le\varepsilon} \frac{L_f(x,r)}{r}. \tag{5.10}$$

Clearly, L_f decreases as ε decreases and

$$\lim_{\varepsilon\downarrow 0} L_f^\varepsilon(x) = L_f(x), \qquad \text{for all } x \in X.$$

Lemma 5.7 ([42, Lemma 7.16]) *Let $f\colon (X_1, d_1) \to (X_2, d_2)$ be a η-quasisymmetry. Let $x_0 \in X$ and $0 < R < \operatorname{diam}(X, d_1)$,. There is a constant C (that depends only on η) such that for all $\varepsilon > 0$, the function CL_f^ε is an upper gradient of the function $u(x) = d_2(f(x), f(x_0))$ in $B(x_0, R)$.*

We introduce the notion of C-approximation to compare balls in different metrics.

Definition 5.8 Let d_1 and d_2 be two metrics on X such that the identity map $\mathrm{Id} : (X, d_1) \to (X, d_2)$ is a η-quasisymmetry. Let $C \ge 1$ be a constant. We say that a ball $B_{d_2}(x_2, r_2)$ is a *C-approximation* of $B_{d_1}(x_1, r_1)$ if

$$\begin{gathered} d_1(x_1, x_2) \le Cr_1, \qquad d_2(x_1, x_2) \le Cr_2 \\ B_{d_1}(x_2, C^{-1}r_1) \subset B_{d_2}(x_2, r_2) \subset B_{d_1}(x_2, Cr_1), \\ B_{d_2}(x_1, C^{-1}r_2) \subset B_{d_1}(x_1, r_1) \subset B_{d_2}(x_1, Cr_2) \end{gathered}$$

By the same argument as Proposition 4.12, we obtain the following comparison of capacities.

Lemma 5.9 *Let $(X, d_i, m_i, \mathcal{E}, \mathcal{F}_i)$, $i = 1, 2$, be two MMD spaces that satisfy PHI(2) such that the identity map $\mathrm{Id}\colon (X, d_1) \to (X, d_2)$ is a quasisymmetry. Let $B_i(x, r)$ denote a open ball of radius r and center x, for $i = 1, 2$. Let $C_1 \ge 1$ and $A_1, A_2 > 1$. There exists $C_2, A_3 > 1$ such that*

$$\operatorname{Cap}(B_1(x_1, r_1), B_1(x_1, A_1r_1)^c) \le C_2 \operatorname{Cap}(B_2(x_2, r_2), B_2(x_2, A_2r_2)^c)$$

for all balls $B_1(x_1, r_1)$ and $B_2(x_2, r_2)$ such that $r_1 < \operatorname{diam}(X, d_1)/A_3$, $r_2 < \operatorname{diam}(X, d_2)/A_3$ and that $B_1(x_1, r_1)$ is a C_1-approximation of $B_2(x_2, r_2)$.

The following is an analogue of [42, Lemma 7.19]

Lemma 5.10 *Let $(X, d_i, m_i, \mathcal{E}, \mathcal{F}_e \cap L^2(m_i))$, $i = 1, 2$ be two MMD spaces that satisfy PHI(2) and are time changes of each other with full quasi-support. Let the identity map $f\colon (X, d_1) \to (X, d_2)$ be an η-quasisymmetry. Then the* *function* L_f^ε *defined in* (5.10) *is in weak* L^2 *for any* $\varepsilon < R/10$ *and for any ball* $B_{d_1}(x_0, R)$, $R < \operatorname{diam}(X, d_1)$. *Furthermore, there exists* $C \geq 1$ *such that* L_f^ε *satisfies the estimate*

$$m_1\left(\left\{x \in B_{d_1}(x_0, R) : L_f^\varepsilon(x) > t\right\}\right) \leq Ct^{-2} m_2\left(B_{d_1}(x_0, R)\right), \quad (5.11)$$

for all $t > 0, 0 < R < \operatorname{diam}(X, d_1), x_0 \in X$. *Here* $C \geq 1$ *depends only on* η *and the constants associated with the MMD spaces* $(X, d_i, m_i, \mathcal{E}, \mathcal{F}_e \cap L^2(m_i))$, $i = 1, 2$.

Proof Let E_t denote the set

$$E_t := \left\{x \in B_{d_1}(x_0, R) : L_f^\varepsilon(x) > t\right\}.$$

Then by the $5B$-covering lemma [40, Theorem 1.2] there exists a countable collection of disjoint balls $B_i = B_{d_1}(x_i, r_i)$, $i \in I$ such that $0 < r_i \leq \varepsilon$,

$$\frac{L_f(x_i, r_i)}{r_i} > t \quad (5.12)$$

and

$$E_t \subset \cup_i 5B_i \subset 2B.$$

Note that the metrics d_1, d_2 are uniformly perfect by Proposition 2.11-(a). Define

$$B_i' := B_{d_2}(x_i, L_f(x_i, r_i)/\eta(1)).$$

Roughly speaking, the balls B_i' in d_2-metric approximate the balls B_i in the d_1-metric for each $i \in I$. More precisely, since f is a η-quasisymmetry and d_1, d_2 are uniformly perfect, there exists $C \geq 1$ such that B_i is a C-approximation of B_i' for all $i \in I$. In particular,

$$C^{-1}B_i' \subset B_i \subset CB_i', \quad C^{-1}B_i \subset B_i' \subset CB_i, \text{ for all } i \in I. \quad (5.13)$$

Since $(X, d_i, m_2, \mathcal{E}, \mathcal{F}_e \cap L^2(m_i)), i = 1, 2$ satisfies PHI(2), there exist $A_1, A_2 > 1$ such that

$$\operatorname{Cap}(B_i, (A_1 B_i)^c) \asymp \frac{m_1(B_i)}{r_i^2},$$

$$\operatorname{Cap}(B_i', (A_2 B_i')^c) \asymp \frac{m_2(B_i')}{L_f(x_i, r_i)^2} \quad \text{for all } i \in I. \tag{5.14}$$

Furthermore, by (5.13) and Lemma 5.9, we have

$$\operatorname{Cap}(B_i, (A_1 B_i)^c) \asymp \operatorname{Cap}(B_i', (A_2 B_i')^c) \quad \text{for all } i \in I. \tag{5.15}$$

We combine the above estimates, to obtain (5.11) as follows:

$$\begin{aligned}
m_1(E_t) &\leq \sum_i m_1(5B_i) \lesssim \sum_i m_1(B_i) \quad (\text{by VD of } (X, d_1, m_1)) \\
&\lesssim \sum_i r_i^2 \operatorname{Cap}(B_i, (A_1 B_i)^c) \quad (\text{by (5.14)}) \\
&\lesssim t^{-2} \sum_i L_f(x_i, r_i)^2 \operatorname{Cap}(B_i, (A_1 B_i)^c) \quad (\text{by (5.12)}) \\
&\lesssim t^{-2} \sum_i L_f(x_i, r_i)^2 \operatorname{Cap}(B_i', (A_2 B_i')^c) \quad (\text{by (5.15)}) \\
&\lesssim t^{-2} \sum_i m_2(B_i') \quad (\text{by (5.14)}) \\
&\lesssim t^{-2} \sum_i m_2(C^{-1} B_i') \quad (\text{by VD of } (X, d_2, m_2)) \\
&\lesssim t^{-2} \sum_i m_2(B_i) \quad (\text{since } C^{-1} B_i' \subset B_i) \\
&\lesssim t^{-2} m_2\left(B_{d_1}(x_0, 2R)\right) \\
&\qquad (\text{since } B_i\text{'s are disjoint and } \cup_i B_i \subset B_{d_1}(x_0, 2R)) \\
&\lesssim t^{-2} m_2\left(B_{d_1}(x_0, R)\right) \quad (\text{by VD of } (X, d_1, m_2)).
\end{aligned}$$

The claimed dependence of the constant C in (5.11) follows from the above argument. □

Corollary 5.11 ([42, Corollary 7.21]) *Let $(X, d_i, m_i, \mathcal{E}, \mathcal{F}_e \cap L^2(m_i))$, $i = 1, 2$ be two MMD spaces that satisfy the assumptions of Lemma 5.10. Let L_f^ε denote the function defined in (5.10). For all $s \in [1, 2)$ and $x_0 \in X$, $0 < \varepsilon < R/10$, $R < \operatorname{diam}(X, d_1)$, the function L_f^ε is in $L^s(B_{d_1}(x_0, R), m_1)$ with*

$$\left(\int_{B_{d_1}(x_0, R)} \left| L_f^\varepsilon \right|^s dm_1 \right)^{1/s} \leq C m_1(B(x_0, R))^{(2-s)/(2s)} m_2\left(B(x_0, R)\right)^{1/s},$$

where C only depends only on s, η and the constants associated with the two MMD spaces. By letting $\varepsilon \downarrow 0$, a similar statement is true for L_f.

Proof of Theorem 2.12 By Proposition 2.11-(b), both m_1 and m_2 are minimal energy dominant measures. Therefore, m_1 and m_2 are mutually absolute continuous. By Proposition 2.11-(a), both d_1 and d_2 are bi-Lipschitz equivalent to intrinsic metrics, and therefore by Lemma 5.6, we may assume that d_1 and d_2 are intrinsic metrics with respect to the symmetric measures m_1 and m_2 respectively.

Let $f\colon (X, d_1) \to (X, d_2)$ denote the identity map, which is an η-quasisymmetry. Then by the Lebesgue–Radon–Nikodym theorem, the volume derivative

$$\mu_f(x) = \lim_{r\downarrow 0} \frac{m_2\left(B_{d_1}(x,r)\right)}{m_1\left(B_{d_1}(x,r)\right)} \tag{5.16}$$

exists and is finite for m_1-almost every $x \in X$. Since $m_2 \ll m_1$, we have $dm_2 = \mu_f\, dm_1$; that is $m_2(E) = \int_E \mu_f\, dm_1$ for all measurable sets E.

Since $(X, d_i, m_i, \mathcal{E}, \mathcal{F} \cap L^2(m_i))$ satisfies PHI(2) for $i = 1, 2$, there exists constants A_1, A_2, C_1, C_2 such that

$$\mathrm{Cap}(B_{d_i}(x,r), B_{d_i}(x, A_i r)^c) \asymp \frac{r^2}{m_i\left(B_{d_i}(x,r)\right)}, \tag{5.17}$$

for all $x \in X$, $r < \mathrm{diam}(X, d_i)/C_i$, $i = 1, 2$. Similar to (5.13), there exists $C \geq 1$ such that for all $r < \mathrm{diam}(X, d_1)$, $x \in X$, $B_{d_1}(x,r)$ is a C-approximation of $B_{d_2}(x, L_f(x,r))$. That is, for all $r < \mathrm{diam}(X, d_1)$, $x \in X$, $B_{d_1}(x,r)$,

$$\begin{aligned} B_{d_2}(x, C^{-1}L_f(x,r)) \subset B_{d_1}(x,r) \subset B_{d_2}(x, CL_f(x,r)), \\ B_{d_1}(x, C^{-1}r) \subset B_{d_2}(x, L_f(x,r)) \subset B_{d_1}(x, Cr). \end{aligned} \tag{5.18}$$

By (5.18), the η-quasisymmetry of f, and the same argument as Proposition 4.12, there exists $C_3 > 1$ such that

$$\begin{aligned} &\mathrm{Cap}(B_{d_1}(x,r), B_{d_1}(x, A_1 r)^c) \\ &\quad \asymp \mathrm{Cap}(B_{d_2}(x, L_f(x,r)), B_{d_2}(x, A_2 L_f(x,r))^c), \end{aligned} \tag{5.19}$$

for all $x \in X$, $r < \mathrm{diam}(X, d_1)/C_3$. Combining (5.17) and (5.19) shows that

$$\frac{L_f(x,r)^2}{r^2} \asymp \frac{m_2(B_{d_2}(x, L_f(x,r)))}{m_1(B_{d_1}(x,r))} \asymp \frac{m_2(B_{d_1}(x,r))}{m_1(B_{d_1}(x,r))}, \tag{5.20}$$

for all $x \in X$, $r < \mathrm{diam}(X, d_1)$. Therefore

$$\mu_f(x) \asymp L_f(x)^2 \quad \text{for almost every } x \in X. \tag{5.21}$$

Let $p \in [1, 2)$ be the constant in Proposition 5.4-(b) so that (X, d_1, m_1) satisfies $(1, p)$-Poincaré inequality. We shall show that L_f satisfies the reverse Hölder inequality

$$\left(\fint_{B_{d_1}(x_0,r)} L_f^2 \, dm_1 \right)^{1/2} \leq C \left(\fint_{B_{d_1}(x_0,r)} L_f^p \, dm_1 \right)^{1/p}, \tag{5.22}$$

for all $x_0 \in X, r < \operatorname{diam}(X, d_1)$. Then by Gehring's lemma [42, Lemma 7.3], Hölder inequality and (5.22), we obtain the following reverse Hölder inequality for the function μ_f: there exists $\varepsilon > 0$ such that

$$\left(\fint_{B_{d_1}(x_0,r)} \mu_f^{1+\varepsilon} \, dm_1 \right)^{1/(1+\varepsilon)} \leq C \fint_{B_{d_1}(x_0,r)} \mu_f \, dm_1,$$

for all $x_0 \in X, r < \operatorname{diam}(X, d_1)$. By the equivalence between reverse Hölder inequality and A_∞-relation as explained in Definition 5.5, it suffices to show (5.22).

Since (X, d_1, m_1) satisfies the $(1, p)$-Poincaré inequality, by Lemma 5.7 CL_f^ε is an upper gradient of $u(x) = d_2(x, x_0)$ in $B_{d_1}(x_0, r)$. Therefore by the Poincaré inequality we have

$$\fint_{B_{d_1}(x_0,K^{-1}r)} \left| u - u_{B_{d_1}(x_0,K^{-1}r)} \right| dm_1 \lesssim r \left(\fint_{B_{d_1}(x_0,r)} (L_f^\varepsilon)^p \, dm_1 \right)^{1/p}.$$

We let $\varepsilon \downarrow 0$ and use Corollary 5.11 and the dominated convergence theorem to obtain

$$\fint_{B_{d_1}(x_0,K^{-1}r)} \left| u - u_{B_{d_1}(x_0,K^{-1}r)} \right| dm_1 \lesssim r \left(\fint_{B_{d_1}(x_0,r)} \left| L_f \right|^p dm_1 \right)^{1/p}. \tag{5.23}$$

By the uniform perfectness of (X, d_1) and the volume doubling property, there exists K_1 such that $m_1 \left(B(x_0, K^{-1}r) \setminus B(x_0, K_1^{-1}K^{-1}r) \right) \gtrsim m_1(B(x_0, r))$. Using the quasisymmetry of f, we obtain

$$\begin{aligned} u_{B_{d_1}(x_0,K^{-1}r)} &= \fint_{B_{d_1}(x_0,K^{-1}r)} d_2(x, x_0) \, m_1(dx) \\ &\geq \frac{1}{m_1(B_{d_1}(x_0, r))} \int_{B(x_0,K^{-1}r) \setminus B(x_0,K_1^{-1}K^{-1}r)} d_2(x, x_0) \, m_1(dx) \end{aligned}$$

$$
\begin{aligned}
&\gtrsim L_f(x_0,r)\frac{m_1\left(B(x_0,K^{-1}r)\setminus B(x_0,K_1^{-1}K^{-1}r)\right)}{m_1(B(x_0,r))}\\
&\geq C_1^{-1}L_f(x_0,r),
\end{aligned}
$$

because

$$
L_f(x_0,r)\lesssim d_2(x,x_0)\quad\text{for all } x\in B(x_0,K^{-1}r)\setminus B(x_0,K_1^{-1}K^{-1}r)
$$

by the quasisymmetry of f and the uniform perfectness of (X,d_1). For sufficiently small $\delta>0$, we similarly have

$$
u(x)=d_2(x,x_0)\leq \eta(\delta K_2)L_f(x_0,r)\leq (2C_1)^{-1}L_f(x_0,r)
$$

for all $x\in B(x_0,\delta K^{-1}r)$. Consequently, using the above estimates and the volume doubling property, we obtain

$$
\begin{aligned}
&⨍_{B_{d_1}(x_0,K^{-1}r)}\left|u-u_{B_{d_1}(x_0,K^{-1}r)}\right|dm_1\\
&\quad\gtrsim ⨍_{B_{d_1}(x_0,\delta K^{-1}r)}\left|u-u_{B_{d_1}(x_0,K^{-1}r)}\right|dm_1\\
&\quad\gtrsim L_f(x_0,r).
\end{aligned}
\tag{5.24}
$$

It follows from the above estimates that

$$
\begin{aligned}
&\left(⨍_{B_{d_1}(x_0,r)}L_f^2\,dm_1\right)^{1/2}\\
&\lesssim\left(⨍_{B_{d_1}(x_0,r)}\mu_f\,dm_1\right)^{1/2}\quad\text{(by (5.21))}\\
&\lesssim\left(\frac{m_2(B_{d_1}(x_0,r))}{m_2\left(B_{d_1}(x_0,r)\right)}\right)^{1/2}\quad\text{(since } dm_2=\mu_f\,dm_1)\\
&\lesssim\frac{L_f(x_0,r)}{r}\quad\text{(by (5.20))}\\
&\lesssim\left(⨍_{B_{d_1}(x_0,r)}\left|L_f\right|^p dm_1\right)^{1/p}\quad\text{(by (5.24) and (5.23)).}
\end{aligned}
$$

This completes the proof of (5.22), and therefore of Theorem 2.12. $\square$

Let $(X, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space that satisfies EHI, where (X, d) is a doubling metric space. If $\mu \in \mathcal{G}(X, d, m, \mathcal{E}, \mathcal{F})$, then by Theorem 2.12

$$\mathcal{G}(X, d, m, \mathcal{E}, \mathcal{F}) \subseteq \{\widetilde{\mu} : \widetilde{\mu} \text{ is } A_\infty\text{-related to } \mu\}. \tag{5.25}$$

One might ask if the inclusion in (5.25) is strict. For the Brownian motion on $\mathbb{R}^n$, the above inclusion is strict if and only if $n \geq 2$ (see Theorem 5.18 and Example 5.14). We need the definition of a maximal semi-metric.

Definition 5.12 A function $r : X \times X \to [0, \infty)$ is said to be a *semi-metric*, if it satisfies all the properties of a metric except possibly the property that $r(x, y) = 0$ implies $x = y$.

Let $h : X \times X \to [0, \infty)$ be an arbitrary function. Then there exists a unique maximal semi-metric $d_h : X \times X \to [0, \infty)$ such that $d_h(x, y) \leq h(x, y)$ for all $x, y \in X$ [22, Lemma 3.1.23]. We call d_h the *maximal semi-metric induced by* h. Equivalently, d_h can be defined as follows. Let $\widetilde{h}(x, y) = \min(h(x, y), h(y, x))$. Then

$$d_h(x, y) = \inf\left\{\sum_{i=0}^{N-1} \widetilde{h}(x_i, x_{i+1}) : N \in \mathbb{N}, x_0 = x, x_N = y\right\}. \tag{5.26}$$

We provide a necessary condition for a measure to be in $\mathcal{G}(X, d, m, \mathcal{E}, \mathcal{F})$. Using this necessary condition, below we obtain examples for which the inclusion (5.25) is strict.

Lemma 5.13 *Let* $(X, d, m, \mathcal{E}, \mathcal{F})$ *satisfy* PHI(γ) *for some* $\gamma \geq 2$. *Let* $\mu \in \mathcal{G}(X, d, m, \mathcal{E}, \mathcal{F})$. *Define*

$$h(x, y) = \sqrt{\frac{\mu(B_d(x, d(x, y)))d(x, y)^\gamma}{m(B_d(x, d(x, y)))}}, \quad \textit{for any } x, y \in X \textit{ with } x \neq y, \tag{5.27}$$

and $h(x, x) = 0$ *for any* $x \in X$. *Let* d_h *denote the maximal semi-metric induced by* h. *Then there exists* $C > 0$ *such that*

$$h(x, y) \leq C d_h(x, y) \quad \textit{for all } x, y \in X.$$

Proof Let $\theta \in \mathcal{J}(X, d)$ be such that the MMD space $(X, \theta, \mu, \mathcal{E}^\mu, \mathcal{F}^\mu)$ satisfies PHI(2). It suffices to show the existence of $C_1 > 0$ such that

$$C_1^{-1}\theta(x, y) \leq h(x, y) \leq C_1\theta(x, y) \quad \text{for all } x, y \in X. \tag{5.28}$$

In particular, due to the triangle inequality for θ, (5.28) implies a similar inequality with $h(x, y)$ replaced by $d_h(x, y)$, which immediately implies that h is comparable to d_h.

By Theorem 4.5, m and μ satisfy VD on (X, d) and (X, θ). By using Proposition 3.2 and VD, there exists $C_2 > 0$ such that

$$C_2^{-1}\mu(B_\theta(x, \theta(x, y))) \leq \mu(B_d(x, d(x, y))) \leq C_2\mu(B_\theta(x, \theta(x, y))) \quad (5.29)$$

for all $x, y \in X$, where $x \neq y$. By an argument similar to the proof of Proposition 4.12 using Lemma 4.11 and Proposition 3.2, there exist $C_3, A > 0$ such that

$$C_3^{-1} \leq \frac{\mathrm{Cap}(B_\theta(x, \theta(x, y)), B_\theta(x, 2\theta(x, y))^c)}{\mathrm{Cap}(B_d(x, d(x, y)), B_d(x, 2d(x, y))^c)} \leq C_3, \quad (5.30)$$

for any pair $x, y \in X$ such that $0 < \theta(x, y) < \mathrm{diam}(X, \theta)/A$. By Theorem 4.5, Lemma 4.11 and Proposition 3.2 and by increasing A if necessary, there exists $C_4 > 0$ such that

$$\begin{aligned}\mathrm{Cap}(B_\theta(x, \theta(x, y)), B_\theta(x, 2\theta(x, y))^c) &\geq C_4^{-1}\frac{\mu(B_\theta(x, \theta(x, y)))}{\theta(x, y)^2},\\ \mathrm{Cap}(B_\theta(x, \theta(x, y)), B_\theta(x, 2\theta(x, y))^c) &\leq C_4\frac{\mu(B_\theta(x, \theta(x, y)))}{\theta(x, y)^2},\end{aligned} \quad (5.31)$$

and

$$\begin{aligned}\mathrm{Cap}(B_d(x, d(x, y)), B_d(x, 2d(x, y))^c) &\geq C_4^{-1}\frac{m(B_d(x, d(x, y)))}{d(x, y)^\gamma},\\ \mathrm{Cap}(B_d(x, d(x, y)), B_d(x, 2d(x, y))^c) &\leq C_4\frac{m(B_d(x, d(x, y)))}{d(x, y)^\gamma},\end{aligned} \quad (5.32)$$

for any pair $x, y \in X$ such that $0 < \theta(x, y) < \mathrm{diam}(X, \theta)/A$. Combining (5.29), (5.30), (5.31) and (5.32) shows that there exists $C_5 > 0$ such that

$$C_5^{-1}\theta(x, y) \leq h(x, y) \leq C_5\theta(x, y), \quad (5.33)$$

for all $x, y \in X$ such that $0 \leq \theta(x, y) < \mathrm{diam}(X, \theta)/A$. Since (X, θ) is uniformly perfect, by replacing y with a closer point $\widetilde{y}$, and using (5.33) and VD, we obtain (5.28). □

Example 5.14 Let $n \geq 2$ and let $(X, d, m, \mathcal{E}, \mathcal{F})$ denote the Dirichlet form corresponding to Brownian motion on $\mathbb{R}^n$. If $w(x) = |x_1|^t$, where $t \in \mathbb{R}$, $x = (x_1, \ldots, x_n)$, then $w\,dm$ is A_∞-related to m if and only if $t > -1$ [83, p. 222, Example (c)]. If $t > 0$, $x = (0, \ldots, 0)$, $y = (0, \ldots, 0, 1)$ and h is as given in (5.27) with $\gamma = 2$, then using (5.26) it is easy to check that $d_h(x, y) = 0$. This can be seen by choosing equally spaced points $x_0, \ldots, x_n$ on the straight line joining x and y and letting $n \to \infty$ in (5.26). Therefore, by Lemma 5.13 we obtain

$$w\,dm \in \{\mu : \mu \text{ is } A_\infty\text{-related to } m\} \setminus \mathcal{G}(X, d, m, \mathcal{E}, \mathcal{F}), \quad \text{for any } n \geq 2, t > 0.$$

In other words, the inclusion in (5.25) is strict for the Brownian motion on $\mathbb{R}^n$, $n \geq 2$.

The above example and Lemma 5.13 illustrate that if a measure is too small in the neighborhood of a curve, then it will fail to be in $\mathcal{G}(X, d, m, \mathcal{E}, \mathcal{F})$. As we will see in Sect. 6.3.2 below, a similar (but more subtle) phenomenon happens in the higher-dimensional Sierpiński gaskets.

We recall the definition of strong A_∞-weights on $\mathbb{R}^n$ introduced by David and Semmes in [29] and show its relevance to the Gaussian uniformization problem for the Brownian motion on $\mathbb{R}^2$. The following definition is a slight reformulation of the one in [29] and the equivalence between the two definitions follows from [83, Lemma 3.1].

Definition 5.15 Let d, m denote the Euclidean metric and Lebesgue measure on $\mathbb{R}^n$, respectively. Let $\mu = w\,dm$ be A_∞-related to m. Define

$$h(x, y) = \left(\mu(B_{x,y})\right)^{1/n},$$

where $B_{x,y}$ is the Euclidean ball with center $z = (x + y)/2$ and radius $d(x, y)/2$. Let d_h denote the maximal semi-metric induced by h. We say that μ is *strong A_∞-related* to m, if there exists $C > 0$ such that

$$d_h(x, y) \geq C^{-1} h(x, y), \quad \text{for all } x, y \in \mathbb{R}^n.$$

The following relates the Gaussian uniformization problem in $\mathbb{R}^2$ in terms of strong A_∞-weights.

Proposition 5.16 *Let $(X, d, m, \mathcal{E}, \mathcal{F})$ denote the MMD space corresponding to the Brownian motion on $\mathbb{R}^2$. Then*

$$\mathcal{G}(X, d, m, \mathcal{E}, \mathcal{F}) = \{\mu : \mu \textit{ is strong } A_\infty\textit{-related to } m\}. \tag{5.34}$$

Proof By Lemma 5.13 and Theorem 2.12, we have the inclusion

$$\mathcal{G}(X, d, m, \mathcal{E}, \mathcal{F}) \subseteq \{\mu : \mu \text{ is strong } A_\infty\text{-related to } m\}. \tag{5.35}$$

For the reverse inclusion, consider a measure $\mu = w\,dm$ that is A_∞-related to m. Since μ is mutually absolutely continuous with respect to m, $\mu \in \mathcal{A}(X, d, m, \mathcal{E}, \mathcal{F})$.

Let θ denote the metric d_h in Definition 5.15. Since μ is A_∞-related to m, μ satisfies VD on (X, d). Since $\theta(x, y)$ is comparable to $\sqrt{\mu(B_{x,y})}$, where $B_{x,y} = B((x + y)/2, d(x, y)/2)$, by the VD and RVD for the measure μ, we obtain that

$$\theta \in \mathcal{J}(X, d). \tag{5.36}$$

By (5.36), Proposition 3.2 and VD, there exists $C_1 > 0$ such that

$$C_1^{-1} r^2 \le \mu(B_\theta(x, r)) \le C_1 r^2 \quad \text{for all } x \in X, r > 0. \tag{5.37}$$

By Lemma 4.11, Proposition 3.2, and (5.36), there exists $C_2 > 0$ such that

$$C_2^{-1} \le \mathrm{Cap}(B_\theta(x, r), B_\theta(x, 2r)^c) \le C_2 \quad \text{for all } x \in X, r > 0. \tag{5.38}$$

By Lemma 4.8, the time-changed MMD space $(X, \theta, \mu, \mathcal{E}^\mu, \mathcal{F}^\mu)$ satisfies EHI. Combining (5.36), (5.37) and (5.38), and using Theorem 4.5, we obtain that $\mu \in \mathcal{G}(X, d, m, \mathcal{E}, \mathcal{F})$. □

Proposition 5.16 along with known results on strong A_∞-related measures leads to many further examples of measures in $\mathcal{G}(X, d, m, \mathcal{E}, \mathcal{F})$ for Brownian motion in $\mathbb{R}^2$. For instance, Bessel potentials can be used to construct strong A_∞-measures [16, Theorem 3.1].

Unlike the case of Brownian motion on $\mathbb{R}^n$ with $n \ge 2$ treated so far, it turns out that the inclusion in (5.25) is an equality for Brownian motion on $\mathbb{R}$, which we prove in the rest of this subsection as a complete answer to the Gaussian uniformization problem for Brownian motion on $\mathbb{R}$. For this purpose, we need the following lemma characterizing A_∞-related weights in terms of a reverse Hölder inequality.

Lemma 5.17 *Let (X, d) be a locally compact metric space and let m be a non-zero Radon measure on (X, d) satisfying VD and such that the function $(0, \infty) \ni r \mapsto m(B(x, r))$ is continuous for each $x \in X$. Then for each $[0, \infty)$-valued $w \in L^1_{\mathrm{loc}}(X, m)$, the following are equivalent:*

(a) $\mu = w\,dm$ is A_∞-related to m.
(b) $w \not\equiv 0$ and there exists $C > 1$ such that the following reverse Hölder inequality holds:

$$\fint_B w\,dm \le C \left(\fint_B \sqrt{w}\,dm \right)^2 \qquad \textit{whenever } B \textit{ is a ball in } (X, d).$$

Proof (b) $\Longrightarrow$ (a): By Gehring's lemma [42, Lemma 7.3], there exist $\varepsilon > 0$ and $C_1 > 0$ such that

$$\left(\fint_B w^{1+\varepsilon}\,dm\right)^{1/(1+\varepsilon)} \le C_1 \left(\fint_B \sqrt{w}\,dm\right)^2 \le \fint_B w\,dm$$

for all balls B in (X, d). By [86, Theorem 18 in Chapter I], μ is A_∞-related to m.
(a) $\Longrightarrow$ (b): By [86, Theorem 18 in Chapter I], there exist $r > 1$ and $C_2 > 1$ such that

$$\left(\fint_B w^r\,dm\right)^{1/r} \le C_2 \fint_B w\,dm \qquad \text{for all balls } B. \tag{5.39}$$

Choose $\theta \in (0, 1)$ such that $\theta r^{-1} + 2(1-\theta) = 1$. By Hölder's inequality and (5.39),

$$\begin{aligned}\fint_B w\,dm &\le \left(\fint_B w^r\,dm\right)^{\theta/r} \left(\fint_B \sqrt{w}\,dm\right)^{2(1-\theta)}\\ &\le C_2^\theta \left(\fint_B w\,dm\right)^{\theta} \left(\fint_B \sqrt{w}\,dm\right)^{2(1-\theta)}\end{aligned}$$

for all balls B. This immediately implies (b) with $C = C_2^{\theta/(1-\theta)}$. $\square$

In the following result, we consider the case of Brownian motion in $\mathbb{R}$; that is $(X, d, m, \mathcal{E}, \mathcal{F})$ is given by $X = \mathbb{R}$, d is the Euclidean distance, m is the Lebesgue measure, $\mathcal{F} = W^{1,2}$ and $\mathcal{E}(f, f) = \int |f'|^2\,dm$.

Theorem 5.18 *Let $(X, d, m, \mathcal{E}, \mathcal{F})$ denote the MMD space corresponding to the Brownian motion on $\mathbb{R}$. Then the family of Gaussian admissible measures is characterized by the reverse Hölder inequality as in Lemma 5.17-(b), i.e.,*

$$\begin{aligned}&\mathcal{G}(X, d, m, \mathcal{E}, \mathcal{F})\\ &= \{\mu \mid \mu \text{ is } A_\infty\text{-related to } m\}\\ &= \{w\,dm \mid w \in L^1_{\mathrm{loc}}(X, m),\ w \text{ is } [0, \infty)\text{-valued and satisfies Lemma 5.17-(b)}\}.\end{aligned}$$

Proof Set

$$\widetilde{\mathcal{G}} = \left\{ g\,dm \;\middle|\; \begin{array}{l} g \not\equiv 0, \text{ there exists } C \ge 1 \text{ such that for all } a < b, \\ (b-a)^{1/2}\left(\int_a^b g\,dm\right)^{1/2} \le C \int_a^b \sqrt{g}\,dm \end{array} \right\}.$$

By Theorem 2.12 and Lemma 5.17, it suffices to show that $\widetilde{\mathcal{G}} \subset \mathcal{G}(X, d, m, \mathcal{E}, \mathcal{F})$. Let $g\,dm \in \widetilde{\mathcal{G}}$. Then consider the measures $\mu_1 = \sqrt{g}\,dm$ and $\mu = g\,dm$. For all $a < b$, an easy calculation shows that

$$d_{\mathrm{int}}^{\mu}(a, b) = \mu_1([a, b]) = \int_a^b \sqrt{g}\,dm. \tag{5.40}$$

Since $\sqrt{g}$ satisfies a reverse Hölder inequality, by [86, Lemma 12 and Theorem 17 in Chapter I], μ_1 is a doubling measure on (X, d). By the correspondence between doubling measures and quasisymmetric maps on $\mathbb{R}$ described in [40, Remark 13.20-(b)] and (5.40), we have

$$d_{\mathrm{int}}^{\mu} \in \mathcal{J}(X, d). \tag{5.41}$$

By the reverse Hölder inequality assumption on $\sqrt{g}$, we have

$$\mu_1([a, b]) \le (b - a)^{1/2} (\mu([a, b]))^{1/2} \le C \mu_1([a, b]).$$

Since μ_1 is a doubling measure on (X, d), the above estimate shows that μ is also a doubling measure on (X, d). Since $d_{\mathrm{int}}^{\mu} \in \mathcal{J}(X, d)$, μ is a doubling measure on $(X, d_{\mathrm{int}}^{\mu})$ by (3.3).

By [86, Theorem 18 of Chapter I] and Lemma 5.17, the measures μ and m are mutually absolutely continuous. This implies that $\mu \in \mathcal{A}(X, d, m, \mathcal{E}, \mathcal{F})$.

By [26, Theorem 1.4], for any interval $I = [a, b]$ and for all $f \in W^{1,2}$, we have a Poincaré inequality

$$\int_I \left(f(x) - \frac{1}{\mu(I)} \int_I f\,d\mu \right)^2 \mu(dx) \le K_{\mu,I}^2 \int_I \left| f'(x) \right|^2 dm(x),$$

where the optimal constant $K_{\mu,I}$ satisfies the two-sided estimate

$$\begin{aligned} K_{\mu,I} \asymp \frac{1}{\mu(I)} \Biggl(&\sup_{a<x<b} \left\{ \mu([x, b])^{1/2} \left(\int_a^x \mu([a, t])^2\,dt \right)^{1/2} \right\} \\ &+ \sup_{a<x<b} \left\{ \mu([a, x])^{1/2} \left(\int_x^b \mu([t, b])^2\,dt \right)^{1/2} \right\} \Biggr). \end{aligned} \tag{5.42}$$

By using the bound $\mu(A) \le \mu(I)$ for all $A \subset I$, (5.42) and reverse Hölder inequality, we have

$$K_{\mu,I}^2 \lesssim \mu(I) m(I) \lesssim \mu_1(I)^2. \tag{5.43}$$

By (5.40), $\mu_1(I)$ is the diameter of I under the intrinsic metric d_{int}^{μ}. Therefore by (5.42) and (5.43), we have the Poincaré inequality PI(2) for the MMD space $(X, d_{\mathrm{int}}^{\mu}, \mu, \mathcal{E}^{\mu}, \mathcal{F}^{\mu})$. By [88, Lemma 1], the MMD space $(X, d_{\mathrm{int}}^{\mu}, \mu, \mathcal{E}^{\mu}, \mathcal{F}^{\mu})$ satisfies CS(2). Since μ is a doubling measure on $(X, d_{\mathrm{int}}^{\mu})$, we have that $\mu \in \mathcal{G}(X, d, m, \mathcal{E}, \mathcal{F})$ by Theorem 4.5. Alternately, the claim that $\mu \in \mathcal{G}(X, d, m, \mathcal{E}, \mathcal{F})$ follows from [89, Theorem 3.5]. □

Remark 5.19 A major obstruction to determining the Gaussian admissible measures for multidimensional Brownian motion in $\mathbb{R}^n, n \geq 2$ is that the intrinsic metric with respect to $\mu = g\,dm$ does not admit a simple description unlike the one-dimensional case where there is a simple formula (5.40). As noted in Example 5.14, the conclusion of Theorem 5.18 fails in higher dimensions.

6 The attainment problem for self-similar sets

In this section, we study the attainment problem, that of whether the infimum in (1.4) defining the conformal walk dimension $d_{\mathrm{cw}} = 2$ is attained, in the case of a self-similar Dirichlet form $(\mathcal{E}, \mathcal{F})$ on a post-critically finite self-similar set K. After introducing the framework of such a Dirichlet form in Sect. 6.1, we prove in Sect. 6.2 that $d_{\mathrm{cw}} = 2$ is attained (if and) only if $(K, \theta_h, \Gamma(h, h), \mathcal{E}, \mathcal{F})$ satisfies PHI(2) for some *harmonic function* $h \in \mathcal{F}$ and a metric θ_h on K quasisymmetric to the resistance metric $R_{\mathcal{E}}$ of $(\mathcal{E}, \mathcal{F})$, where $\Gamma(h, h)$ denotes the energy measure of h associated with $(\mathcal{E}, \mathcal{F})$. Then in Sect. 6.3 we present several examples, all of which are shown NOT to attain $d_{\mathrm{cw}} = 2$ except for the two-dimensional standard Sierpiński gasket, which is known to attain $d_{\mathrm{cw}} = 2$ by the results in [52,62] as discussed in Theorem 6.33 and its proof below.

The restriction of the framework to post-critically finite self-similar sets is mainly for the sake of simplicity. In fact, all the results in Sect. 6.2 can be verified, with just slight modifications in the proofs, also for the canonical self-similar Dirichlet form on any *generalized Sierpiński carpet* introduced in Sect. 6.4, which forms essentially the only class of examples of infinitely ramified self-similar fractals where the theory of a canonical self-similar Dirichlet form has been established. We treat the case of generalized Sierpiński carpets in Sect. 6.4 and explain what changes are needed for the arguments in Sects. 6.1 and 6.2 to go through in this case.

6.1 Preliminaries

In this subsection, we first introduce our framework of a post-critically finite self-similar set and a self-similar Dirichlet form on it, for which we mainly follow the presentation of [54, Section 3], and then present preliminary facts.

Let us start with the standard notions concerning self-similar sets. We refer to [60, Chapter 1] for details. Throughout this subsection, we fix a compact metrizable topological space K, a finite set S with $\#S \geq 2$ and a continuous injective map $F_i \colon K \to K$ for each $i \in S$. We set $\mathcal{L} := (K, S, \{F_i\}_{i\in S})$.

Definition 6.1 (1) Let $W_0 := \{\emptyset\}$, where $\emptyset$ is an element called the *empty word*, let $W_n := S^n = \{w_1 \dots w_n \mid w_i \in S \text{ for } i \in \{1, \dots, n\}\}$ for $n \in \mathbb{N}$ and let $W_* := \bigcup_{n\in\mathbb{N}\cup\{0\}} W_n$. For $w \in W_*$, the unique $n \in \mathbb{N}\cup\{0\}$ with $w \in W_n$ is denoted by $|w|$ and called the *length of* w. For $i \in S$ and $n \in \mathbb{N}\cup\{0\}$ we write $i^n := i \dots i \in W_n$.

(2) We set $\Sigma := S^{\mathbb{N}} = \{\omega_1\omega_2\omega_3 \dots \mid \omega_i \in S \text{ for } i \in \mathbb{N}\}$, which is always equipped with the product topology of the discrete topology on S, and define the *shift map* $\sigma \colon \Sigma \to \Sigma$ by $\sigma(\omega_1\omega_2\omega_3 \dots) := \omega_2\omega_3\omega_4 \dots$. For $i \in S$ we define $\sigma_i \colon \Sigma \to \Sigma$ by $\sigma_i(\omega_1\omega_2\omega_3 \dots) := i\omega_1\omega_2\omega_3 \dots$. For $\omega = \omega_1\omega_2\omega_3 \dots \in \Sigma$ and $n \in \mathbb{N}\cup\{0\}$, we write $[\omega]_n := \omega_1 \dots \omega_n \in W_n$.

(3) For $w = w_1 \dots w_n \in W_*$, we set $F_w := F_{w_1} \circ \cdots \circ F_{w_n}$ ($F_\emptyset := \mathrm{id}_K$), $K_w := F_w(K)$, $\sigma_w := \sigma_{w_1} \circ \cdots \circ \sigma_{w_n}$ ($\sigma_\emptyset := \mathrm{id}_\Sigma$) and $\Sigma_w := \sigma_w(\Sigma)$, and if $w \neq \emptyset$ then $w^\infty \in \Sigma$ is defined by $w^\infty := www\dots$ in the natural manner.

Definition 6.2 $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$ is called a *self-similar structure* if and only if there exists a continuous surjective map $\pi : \Sigma \to K$ such that $F_i \circ \pi = \pi \circ \sigma_i$ for any $i \in S$. Note that such π, if it exists, is unique and satisfies $\{\pi(\omega)\} = \bigcap_{n\in\mathbb{N}} K_{[\omega]_n}$ for any $\omega \in \Sigma$.

In the rest of this subsection we always assume that $\mathcal{L}$ is a self-similar structure.

Definition 6.3 (1) We define the *critical set* $\mathcal{C}_\mathcal{L}$ and the *post-critical set* $\mathcal{P}_\mathcal{L}$ of $\mathcal{L}$ by

$$\mathcal{C}_\mathcal{L} := \pi^{-1}\Bigl(\bigcup_{i,j\in S,\, i\neq j} K_i \cap K_j\Bigr) \qquad \text{and} \qquad \mathcal{P}_\mathcal{L} := \bigcup_{n\in\mathbb{N}} \sigma^n(\mathcal{C}_\mathcal{L}). \tag{6.1}$$

$\mathcal{L}$ is called *post-critically finite*, or *p.-c.f.* for short, if and only if $\mathcal{P}_\mathcal{L}$ is a finite set.

(2) We set $V_0 := \pi(\mathcal{P}_\mathcal{L})$, $V_n := \bigcup_{w\in W_n} F_w(V_0)$ for $n \in \mathbb{N}$ and $V_* := \bigcup_{n\in\mathbb{N}\cup\{0\}} V_n$.

V_0 should be considered as the *"boundary"* of the self-similar set K; indeed, $K_w \cap K_v = F_w(V_0) \cap F_v(V_0)$ for any $w, v \in W_*$ with $\Sigma_w \cap \Sigma_v = \emptyset$ by [60, Proposition 1.3.5-(2)]. According to [60, Lemma 1.3.11], $V_{n-1} \subset V_n$ for any $n \in \mathbb{N}$, and if $V_0 \neq \emptyset$ then V_* is dense in K. Also note that by [60, Theorem 1.6.2], K is connected if and only if for any $i, j \in S$ there exist $n \in \mathbb{N}$ and

$\{i_k\}_{k=0}^n \subset S$ with $i_0 = i$ and $i_n = j$ such that $K_{i_{k-1}} \cap K_{i_k} \neq \emptyset$ for any $k \in \{1, \ldots, n\}$, and if K is connected then it is arcwise connected.

In the remainder of this subsection our self-similar structure $\mathcal{L} = (K, S, \{F_i\}_{i \in S})$ is always assumed to be post-critically finite with K connected, so that $2 \leq \#V_0 < \infty$, $K \neq \overline{V_0} = V_0$ and V_* is countably infinite and dense in K.

Next we briefly recall the construction and basic properties of a self-similar Dirichlet form on such $\mathcal{L}$; see [60, Chapter 3] for details. Let $D = (D_{pq})_{p,q \in V_0}$ be a real symmetric matrix of size $\#V_0$ (which we also regard as a linear operator on $\mathbb{R}^{V_0}$) such that

(D1) $\{u \in \mathbb{R}^{V_0} \mid Du = 0\} = \mathbb{R}\mathbb{1}_{V_0}$,
(D2) $D_{pq} \geq 0$ for any $p, q \in V_0$ with $p \neq q$.

We define

$$\begin{aligned} \mathcal{E}^{(0)}(u, v) &:= -\sum_{p,q \in V_0} D_{pq} u(q) v(p) \\ &= \frac{1}{2} \sum_{p,q \in V_0} D_{pq}(u(p) - u(q))(v(p) - v(q)) \end{aligned} \tag{6.2}$$

for $u, v \in \mathbb{R}^{V_0}$, so that $(\mathcal{E}^{(0)}, \mathbb{R}^{V_0})$ is a Dirichlet form on $L^2(V_0, \#)$. Furthermore let $\mathbf{r} = (r_i)_{i \in S} \in (0, \infty)^S$ and define

$$\mathcal{E}^{(n)}(u, v) := \sum_{w \in W_n} \frac{1}{r_w} \mathcal{E}^{(0)}(u \circ F_w|_{V_0}, v \circ F_w|_{V_0}), \quad u, v \in \mathbb{R}^{V_n} \tag{6.3}$$

for each $n \in \mathbb{N}$, where $r_w := r_{w_1} r_{w_2} \ldots r_{w_n}$ for $w = w_1 w_2 \ldots w_n \in W_n$ ($r_\emptyset := 1$).

Definition 6.4 The pair $(D, \mathbf{r})$ of a real symmetric matrix $D = (D_{pq})_{p,q \in V_0}$ of size $\#V_0$ with the properties (D1) and (D2) and $\mathbf{r} = (r_i)_{i \in S} \in (0, \infty)^S$ is called a *harmonic structure* on $\mathcal{L}$ if and only if $\mathcal{E}^{(0)}(u, u) = \inf_{v \in \mathbb{R}^{V_1},\, v|_{V_0} = u} \mathcal{E}^{(1)}(v, v)$ for any $u \in \mathbb{R}^{V_0}$; note that then

$$\mathcal{E}^{(n_1)}(u, u) = \min_{v \in \mathbb{R}^{V_{n_2}},\, v|_{V_{n_1}} = u} \mathcal{E}^{(n_2)}(v, v) \tag{6.4}$$

for any $n_1, n_2 \in \mathbb{N} \cup \{0\}$ with $n_1 \leq n_2$ and any $u \in \mathbb{R}^{V_{n_1}}$ by [60, Proposition 3.1.3]. If $\mathbf{r} \in (0, 1)^S$ in addition, then $(D, \mathbf{r})$ is called *regular*.

In the rest of this subsection, we assume that $(D, \mathbf{r})$ is a regular harmonic structure on $\mathcal{L}$. In this case, $\{\mathcal{E}^{(n)}(u|_{V_n}, u|_{V_n})\}_{n \in \mathbb{N} \cup \{0\}}$ is non-decreasing and hence has the limit in $[0, \infty]$ for any $u \in C(K)$. Then we define a linear subspace $\mathcal{F}$ of $C(K)$ and a non-negative definite symmetric bilinear form $\mathcal{E}\colon \mathcal{F} \times \mathcal{F} \to \mathbb{R}$ by

$$\mathcal{F} := \{u \in C(K) \mid \lim_{n\to\infty} \mathcal{E}^{(n)}(u|_{V_n}, u|_{V_n}) < \infty\}, \tag{6.5}$$

$$\mathcal{E}(u, v) := \lim_{n\to\infty} \mathcal{E}^{(n)}(u|_{V_n}, v|_{V_n}) \in \mathbb{R}, \quad u, v \in \mathcal{F}, \tag{6.6}$$

so that $(\mathcal{E}, \mathcal{F})$ is easily seen to possess the following self-similarity properties (note that $\mathcal{F} \cap C(K) = \mathcal{F}$ in the present setting):

$$\mathcal{F} \cap C(K) = \{u \in C(K) \mid u \circ F_i \in \mathcal{F} \text{ for any } i \in S\}, \tag{6.7}$$

$$\mathcal{E}(u, v) = \sum_{i\in S} \frac{1}{r_i} \mathcal{E}(u \circ F_i, v \circ F_i), \quad u, v \in \mathcal{F} \cap C(K). \tag{6.8}$$

By [60, Proposition 2.2.4, Lemma 2.2.5, Theorem 2.2.6, Lemma 2.3.9, Theorems 2.3.10 and 3.3.4], $(\mathcal{E}, \mathcal{F})$ is a resistance form on K and its resistance metric $R_{\mathcal{E}}\colon K \times K \to [0, \infty)$ is a metric on K compatible with the original topology of K; here $(\mathcal{E}, \mathcal{F})$ being a *resistance form* on K means that it has the following properties (see [60, Definition 2.3.1] or [64, Definition 3.1]):

(RF1) $\{u \in \mathcal{F} \mid \mathcal{E}(u, u) = 0\} = \mathbb{R}\mathbb{1}_K$.
(RF2) $(\mathcal{F}/\mathbb{R}\mathbb{1}_K, \mathcal{E})$ is a Hilbert space.
(RF3) $\{u|_V \mid u \in \mathcal{F}\} = \mathbb{R}^V$ for any non-empty finite subset V of K.
(RF4) $R_{\mathcal{E}}(x, y) := \sup_{u\in\mathcal{F}\setminus\mathbb{R}\mathbb{1}_K} |u(x) - u(y)|^2/\mathcal{E}(u, u) < \infty$ for any $x, y \in K$.
(RF5) $u^+ \wedge 1 \in \mathcal{F}$ and $\mathcal{E}(u^+ \wedge 1, u^+ \wedge 1) \leq \mathcal{E}(u, u)$ for any $u \in \mathcal{F}$.

See [60, Chapter 2] and [64, Part 1] for further details of resistance forms.

In the present framework, the notion of harmonic functions is defined as follows.

Definition 6.5 Let $n \in \mathbb{N} \cup \{0\}$. A continuous function $h \in C(K)$ is called *$\mathcal{E}$-harmonic* on $K \setminus V_n$, or *n-harmonic* for short, if and only if $h \in \mathcal{F}$ and

$$\begin{aligned} \mathcal{E}(h, h) &= \inf_{v\in\mathcal{F},\, v|_{V_n}=h|_{V_n}} \mathcal{E}(v, v), \\ \text{or equivalently,} \quad \mathcal{E}(h, v) &= 0 \ \text{ for any } v \in \mathcal{F}^{K\setminus V_n}, \end{aligned} \tag{6.9}$$

where $\mathcal{F}^{K\setminus V_n} := \{u \in \mathcal{F} \mid u|_{V_n} = 0\}$. We set $\mathcal{H}_n := \{h \in C(K) \mid h \text{ is } n\text{-harmonic}\}$.

It is obvious that $\mathcal{H}_n$ is a linear subspace of $\mathcal{F}$ and $\mathbb{R}\mathbb{1}_K \subset \mathcal{H}_n \subset \mathcal{H}_{n+1}$ for any $n \in \mathbb{N} \cup \{0\}$. Moreover, we easily have the following proposition by [60, Lemma 2.2.2 and Theorem 3.2.4], (6.3), (6.4), (6.7) and (6.8).

Proposition 6.6 *Let $n \in \mathbb{N} \cup \{0\}$.*

(1) For each $u \in \mathbb{R}^{V_n}$ there exists a unique $H_n(u) \in \mathcal{H}_n$ such that $H_n(u)|_{V_n} = u$. Moreover, $H_n \colon \mathbb{R}^{V_n} \to \mathcal{H}_n$ is linear (and hence it is a linear isomorphism).

(2) It holds that

$$\begin{aligned} \mathcal{H}_n &= \{h \in \mathcal{F} \mid \mathcal{E}(h,h) = \mathcal{E}^{(n)}(h|_{V_n}, h|_{V_n})\} && (6.10)\\ &= \{h \in C(K) \mid h \circ F_w \in \mathcal{H}_0 \text{ for any } w \in W_n\}. && (6.11) \end{aligned}$$

In particular, for each $w \in W_*$, a linear map $F_w^* \colon \mathcal{H}_0 \to \mathcal{H}_0$ is defined by $F_w^* h := h \circ F_w$.

Now we equip K with a measure to turn $(\mathcal{E}, \mathcal{F})$ into a Dirichlet form. Indeed, we have the following proposition.

Proposition 6.7 *Let μ be a Radon measure on K with full support. Then $(\mathcal{E}, \mathcal{F})$ is an irreducible, strongly local, regular symmetric Dirichlet form on $L^2(K,\mu)$, and its extended Dirichlet space $\mathcal{F}_e$ coincides with $\mathcal{F}$. Moreover, the capacity Cap_1^{μ} associated with $(K, R_{\mathcal{E}}, \mu, \mathcal{E}, \mathcal{F})$ satisfies $\inf_{x \in K} \mathrm{Cap}_1^{\mu}(\{x\}) > 0$, and in particular* (recall Definition 2.8)

$$\mathcal{A}(K, R_{\mathcal{E}}, \mu, \mathcal{E}, \mathcal{F}) = \{\nu \mid \nu \text{ is a Radon measure on } K \text{ with full support}\}. \tag{6.12}$$

Proof This proposition is well known to experts on Dirichlet forms on fractals, but we include a complete proof of it for the reader's convenience. $(\mathcal{E}, \mathcal{F})$ is a regular symmetric Dirichlet form on $L^2(K,\mu)$ by [64, Corollary 6.4 and Theorem 9.4], strongly local by the same argument as [45, Proof of Lemma 3.12] on the basis of (6.7), (6.8) and $\mathcal{E}(\mathbb{1}_K, \mathbb{1}_K) = 0$, and irreducible by (RF1) above and [24, Theorem 2.1.11]. The equality $\mathcal{F}_e = \mathcal{F}$ is immediate from (RF1), (RF2) and (RF4). We also easily see from (RF4), $\mathrm{diam}_{R_{\mathcal{E}}}(K) < \infty$ and $\mu(K) < \infty$ that $\inf_{x \in K} \mathrm{Cap}_1^{\mu}(\{x\}) > 0$, so that a subset of K is quasi-closed with respect to $(K, R_{\mathcal{E}}, \mu, \mathcal{E}, \mathcal{F})$ if and only if it is closed in K. In particular, any Radon measure ν on K is smooth with respect to $(K, R_{\mathcal{E}}, \mu, \mathcal{E}, \mathcal{F})$ and ν having full quasi-support with respect to $(K, R_{\mathcal{E}}, \mu, \mathcal{E}, \mathcal{F})$ means that the only closed subset F of K with $\nu(K \setminus F) = 0$ is $F = K$, which together imply (6.12). □

Let $d_{\mathrm{H}} \in (0,\infty)$ be such that $\sum_{i \in S} r_i^{d_{\mathrm{H}}} = 1$, so that $d_{\mathrm{H}} \geq 1$ since

$$\max_{x,y \in V_0} R_{\mathcal{E}}(x,y) \leq \sum_{i \in S} \max_{x,y \in V_0} R_{\mathcal{E}}(F_i(x), F_i(y)) \leq \left(\sum_{i \in S} r_i\right) \max_{x,y \in V_0} R_{\mathcal{E}}(x,y) \tag{6.13}$$

by the connectedness of K, [60, Theorem 1.6.2 and Lemma 3.3.5] and hence $\sum_{i\in S} r_i \geq 1 = \sum_{i\in S} r_i^{d_{\mathrm{H}}}$. Let m be the *self-similar measure* on $\mathcal{L}$ with weight $(r_i^{d_{\mathrm{H}}})_{i\in S}$, i.e., the unique Borel measure on K such that $m(K_w) = r_w^{d_{\mathrm{H}}}$ for any $w \in W_*$. The measure m could be considered as the "uniform distribution" on $\mathcal{L}$, and it is the most typical choice of the reference measure μ for $(\mathcal{E}, \mathcal{F})$. It is well known that $(K, R_{\mathcal{E}}, m, \mathcal{E}, \mathcal{F})$ satisfies PHI($d_{\mathrm{H}} + 1$); more precisely, the following lemma and proposition hold.

Lemma 6.8 *There exist $c_1, c_2 \in (0, \infty)$ such that for any $(x, s) \in K \times (0, \mathrm{diam}_{R_{\mathcal{E}}}(K)]$,*

$$c_1 s^{d_{\mathrm{H}}} \leq m(B_{R_{\mathcal{E}}}(x, s)) \leq c_2 s^{d_{\mathrm{H}}}. \tag{6.14}$$

Proof This is immediate from Lemma 6.14 below and [60, Lemma 4.2.3]. □

Proposition 6.9 *$(K, R_{\mathcal{E}}, m, \mathcal{E}, \mathcal{F})$ satisfies* PHI($d_{\mathrm{H}} + 1$).

Proof Lemma 6.8 and [64, Theorem 15.10] together imply that $(K, R_{\mathcal{E}}, m, \mathcal{E}, \mathcal{F})$ satisfies HKE($d_{\mathrm{H}}+1$) as well as VD and RVD, and therefore it also satisfies PHI($d_{\mathrm{H}} + 1$) by Theorem 4.5. □

We conclude this subsection with the following Proposition, which is essentially due to Kigami [63,64] and gives a simple equivalent condition for the validity of PHI(β) after quasisymmetric change of the metric and time change.

Proposition 6.10 *Let $\theta \in \mathcal{J}(K, R_{\mathcal{E}})$* (recall (1.2))*, let μ be a Radon measure on K with full support and let $\beta \in (1, \infty)$. Then the following conditions are equivalent:*

(a) $(K, \theta, \mu, \mathcal{E}, \mathcal{F})$ satisfies PHI(β).
(b) There exists $C \in (1, \infty)$ such that for any $w \in W_*$,

$$C^{-1}(\mathrm{diam}_\theta(K_w))^\beta \leq r_w \mu(K_w) \leq C(\mathrm{diam}_\theta(K_w))^\beta. \tag{6.15}$$

Moreover, if either of these conditions holds, then $\mu(F_w(V_0)) = 0$ for any $w \in W_*$ and $\mu(\{x\}) = 0$ for any $x \in K$.

The rest of this subsection is devoted to the proof of Proposition 6.10, which requires the following lemmas and definitions.

Lemma 6.11 ([60, Lemma 3.3.5], [61, Theorem A.1]) *There exists $c_{R_{\mathcal{E}}} \in (0, 1]$ such that for any $w \in W_*$ and any $x, y \in K$,*

$$c_{R_{\mathcal{E}}} r_w R_{\mathcal{E}}(x, y) \leq R_{\mathcal{E}}(F_w(x), F_w(y)) \leq r_w R_{\mathcal{E}}(x, y). \tag{6.16}$$

Definition 6.12 (1) Let $w, v \in W_*$, $w = w_1 \dots w_{n_1}$, $v = v_1 \dots v_{n_2}$. We define $wv \in W_*$ by $wv := w_1 \dots w_{n_1} v_1 \dots v_{n_2}$ ($w\emptyset := w$, $\emptyset v := v$). We write $w \le v$ if and only if $w = v\tau$ for some $\tau \in W_*$; note that $\Sigma_w \cap \Sigma_v = \emptyset$ if and only if neither $w \le v$ nor $v \le w$.
(2) A finite subset Λ of W_* is called a *partition* of Σ if and only if $\Sigma_w \cap \Sigma_v = \emptyset$ for any $w, v \in \Lambda$ with $w \ne v$ and $\Sigma = \bigcup_{w\in\Lambda} \Sigma_w$.
(3) Let Λ_1, Λ_2 be partitions of Σ. We say that Λ_1 is a *refinement of* Λ_2, and write $\Lambda_1 \le \Lambda_2$, if and only if for each $w^1 \in \Lambda_1$ there exists $w^2 \in \Lambda_2$ such that $w^1 \le w^2$.

Definition 6.13 (1) We define $\Lambda_1 := \{\emptyset\}$,

$$\Lambda_s := \{w \mid w = w_1 \dots w_n \in W_* \setminus \{\emptyset\}, r_{w_1\dots w_{n-1}} > s \ge r_w\} \tag{6.17}$$

for each $s \in (0, 1)$, and $\mathscr{S} := \{\Lambda_s\}_{s\in(0,1]}$. We call $\mathscr{S}$ the *scale* on Σ associated with $\mathbf{r}$.
(2) For each $(s, x) \in (0, 1] \times K$, we define $\Lambda_{s,x} := \{w \in \Lambda_s \mid x \in K_w\}$, $K_s(x) := \bigcup_{w\in\Lambda_{s,x}} K_w$, $\Lambda^1_{s,x} := \{w \in \Lambda_s \mid K_w \cap K_s(x) \ne \emptyset\}$ and $U_s(x) := \bigcup_{w\in\Lambda^1_{s,x}} K_w$.

Clearly $\lim_{s\downarrow 0} \min\{|w| \mid w \in \Lambda_s\} = \infty$, and it is easy to see that Λ_s is a partition of Σ for any $s \in (0, 1]$ and that $\Lambda_{s_1} \le \Lambda_{s_2}$ for any $s_1, s_2 \in (0, 1]$ with $s_1 \le s_2$. These facts together with [60, Proposition 1.3.6] imply that for any $x \in K$, each of $\{K_s(x)\}_{s\in(0,1]}$ and $\{U_s(x)\}_{s\in(0,1]}$ is non-decreasing in s and forms a fundamental system of neighborhoods of x in K. Moreover, $\{U_s(x)\}_{(s,x)\in(0,1]\times K}$ can be used as a replacement for the metric balls $\{B_{R_\mathcal{E}}(x, s)\}_{(x,s)\in K\times(0,\mathrm{diam}(K,R_\mathcal{E})]}$ in $(K, R_\mathcal{E})$ by virtue of the following lemma.

Lemma 6.14 *There exist $\alpha_1, \alpha_2 \in (0, \infty)$ such that for any $(s, x) \in (0, 1] \times K$,*

$$B_{R_\mathcal{E}}(x, \alpha_1 s) \subset U_s(x) \subset B_{R_\mathcal{E}}(x, \alpha_2 s). \tag{6.18}$$

Proof This is mentioned in [53, Subsection 4.1], but we include a complete proof of it for the reader's convenience. By the upper inequality in (6.16) we have $\mathrm{diam}_{R_\mathcal{E}}(K_w) \le r_w \,\mathrm{diam}_{R_\mathcal{E}}(K)$ for any $w \in W_*$, which implies the latter inclusion in (6.18) with $\alpha_2 \in (2\,\mathrm{diam}_{R_\mathcal{E}}(K), \infty)$ arbitrary. On the other hand, by [60, Proof of Lemma 4.2.4] there exists $\alpha_1 \in (0, \infty)$ such that $R_\mathcal{E}(x, y) \ge \alpha_1 s$ for any $s \in (0, 1]$, any $w, v \in \Lambda_s$ with $K_w \cap K_v = \emptyset$ and any $(x, y) \in K_w \times K_v$, which yields the former inclusion in (6.18). $\square$

Proof of Proposition 6.10 This equivalence can be easily concluded by combining Theorem 4.5 and results in [63,64], as follows. First, by Theorem 4.5 and [64, Theorem 15.10], under $\theta \in \mathcal{J}(K, R_\mathcal{E})$, (a) is equivalent to the following condition (c):

(c) (K, θ, μ) is VD and there exists $C \in (1, \infty)$ such that for any $x, y \in K$ with $x \neq y$,

$$C^{-1}\theta(x, y)^{\beta} \leq R_{\mathcal{E}}(x, y)\mu\bigl(B_{\theta}(x, \theta(x, y))\bigr) \leq C\theta(x, y)^{\beta}. \quad (6.19)$$

Next, by $\theta \in \mathcal{J}(K, R_{\mathcal{E}})$, (1.3) and (3.3) with $(d_1, d_2) \in \{(R_{\mathcal{E}}, \theta), (\theta, R_{\mathcal{E}})\}$, (K, θ, μ) is VD if and only if $(K, R_{\mathcal{E}}, \mu)$ is VD, which in turn is, by Lemma 6.14 and the compactness of K, equivalent to the existence of $C \in (1, \infty)$ such that

$$\mu(U_s(x)) \leq C\mu(U_{s/2}(x)) \qquad \text{for any } (s, x) \in (0, 1] \times K. \quad (6.20)$$

Then by [63, Theorem 1.3.5] and the fact that $\mathscr{S}$ is *locally finite* with respect to $\mathcal{L}$, i.e.,

$$\sup_{(s,x)\in(0,1]\times K} \#\Lambda^{1}_{s,x} < \infty \quad (6.21)$$

by [60, Lemma 4.2.3] and [63, Lemma 1.3.6], we have (6.20) if and only if there exists $C \in (1, \infty)$ such that the following hold:

$$\mu(K_{wj}) \geq C^{-1}\mu(K_w) \quad \text{for any } (w, j) \in W_* \times S, \quad (6.22)$$

$$\mu(K_w) \leq C\mu(K_v) \quad \text{for any } s \in (0, 1] \text{ and any } w, v \in \Lambda_s \text{ with } K_w \cap K_v \neq \emptyset. \quad (6.23)$$

Moreover, by [63, Theorem 1.2.4], (6.22) implies that

$$\mu(F_w(V_0)) = 0 = \mu(\{x\}) \qquad \text{for any } w \in W_* \text{ and any } x \in K. \quad (6.24)$$

(We remark that (6.24) is part of the assumptions of [63, Theorem 1.3.5] but can be dropped; indeed, even without assuming (6.24), [63, Proofs of Theorems 1.3.10 and 1.3.11] show that any one of the three conditions [63, Theorem 1.3.5-(1),(2),(3)] implies (6.22), from which (6.24) also follows by [63, Theorem 1.2.4].)

On the other hand, since the quasisymmetry of $R_{\mathcal{E}}$ to θ yields $\delta_1, \delta_2 \in (0, \infty)$ such that $B_{\theta}(x, \delta_1\theta(x, y)) \subset B_{R_{\mathcal{E}}}(x, R_{\mathcal{E}}(x, y)) \subset B_{\theta}(x, \delta_2\theta(x, y))$ for any $x, y \in K$ with $x \neq y$ by (3.3), under VD of (K, θ, μ) and $(K, R_{\mathcal{E}}, \mu)$ we have (6.19) if and only if there exists $C \in (0, \infty)$ such that for any $x, y \in K$ with $x \neq y$,

$$C^{-1}\theta(x, y)^{\beta} \leq R_{\mathcal{E}}(x, y)\mu\bigl(B_{R_{\mathcal{E}}}(x, R_{\mathcal{E}}(x, y))\bigr) \leq C\theta(x, y)^{\beta}. \quad (6.25)$$

Therefore (c) is equivalent to the following condition (d):

(d) There exists $C \in (1, \infty)$ such that (6.22), (6.23) and (6.25) hold.

Thus it remains to show that (d) is equivalent to (b). Indeed, let $w \in W_*$ and take $x, y \in K_w$ with the property $\operatorname{diam}_{R_\mathcal{E}}(K_w) = R_\mathcal{E}(x, y)$, so that $w \in \Lambda_{r_w}$, $R_\mathcal{E}(x, y)/\operatorname{diam}_{R_\mathcal{E}}(K) \in [c_{R_\mathcal{E}} r_w, r_w]$ by Lemma 6.11, and $\theta(x, y)^\beta = (\operatorname{diam}_\theta(\{x, y\}))^\beta \asymp (\operatorname{diam}_\theta(K_w))^\beta$ by $\theta \in \mathcal{J}(K, R_\mathcal{E})$ and (3.7). If (d) holds, then since $(K, R_\mathcal{E}, \mu)$ is VD we easily see from Lemma 6.14, (6.21) and (6.23) that

$$\mu\bigl(B_{R_\mathcal{E}}(x, \operatorname{diam}_{R_\mathcal{E}}(K_w))\bigr) = \mu\bigl(B_{R_\mathcal{E}}(x, R_\mathcal{E}(x, y))\bigr) \asymp \mu(K_w) \tag{6.26}$$

and hence (6.25) implies (b). Conversely suppose that (b) holds. Then for any $j \in S$, Lemma 6.11 yields $\operatorname{diam}_{R_\mathcal{E}}(K_{wj})/\operatorname{diam}_{R_\mathcal{E}}(K_w) \in [c_{R_\mathcal{E}} \min_{k \in S} r_k, c_{R_\mathcal{E}}^{-1} \max_{k \in S} r_k]$, hence

$$(\operatorname{diam}_\theta(K_{wj}))^\beta \asymp (\operatorname{diam}_\theta(K_w))^\beta \tag{6.27}$$

by $\theta \in \mathcal{J}(K, R_\mathcal{E})$ and (3.7), and therefore (6.15) implies $\mu(K_{wj}) \asymp \mu(K_w)$, i.e., (6.22) holds. Also for any $s \in (0, 1]$ and any $v, \tau \in \Lambda_s$ with $K_v \cap K_\tau \neq \emptyset$, $\operatorname{diam}_{R_\mathcal{E}}(K_v) \asymp \operatorname{diam}_{R_\mathcal{E}}(K_v \cup K_\tau) \asymp \operatorname{diam}_{R_\mathcal{E}}(K_\tau)$ by Lemma 6.11 and hence

$$(\operatorname{diam}_\theta(K_v))^\beta \asymp (\operatorname{diam}_\theta(K_v \cup K_\tau))^\beta \asymp (\operatorname{diam}_\theta(K_\tau))^\beta \tag{6.28}$$

by $\theta \in \mathcal{J}(K, R_\mathcal{E})$ and (3.7), which together with (6.15) implies $\mu(K_v) \asymp \mu(K_\tau)$, proving (6.23). In particular, $(K, R_\mathcal{E}, \mu)$ is VD, and now it follows from Lemma 6.14, (6.21) and (6.23) that (6.26) holds, which together with (6.15) yields (6.25), proving (d). □

6.2 A necessary condition: attainment by the energy measure of some harmonic function

Throughout this subsection, we assume that $\mathcal{L} = (K, S, \{F_i\}_{i \in S})$ is a post-critically finite self-similar structure with $\#S \geq 2$ and K connected and that $(D, \mathbf{r})$ is a regular harmonic structure on $\mathcal{L}$, and we follow the notation introduced in Sect. 6.1.

Proposition 6.10 with $\beta = 2$ justifies introducing the following set of pairs of metrics and Borel probability measures on K. Recall Definition 6.5 for $\mathcal{H}_0$.

Definition 6.15 We set

$$\mathrm{Homeo}^+ := \{\eta \mid \eta \colon [0, \infty) \to [0, \infty),\ \eta \text{ is a homeomorphism}\}, \tag{6.29}$$
$$\mathcal{P}(K) := \{\mu \mid \mu \text{ is a Borel probability measure on } K\}, \tag{6.30}$$

which is equipped with the topology of weak convergence, and for each $(\eta, C) \in \mathrm{Homeo}^+ \times (1, \infty)$ we define

$$\begin{aligned}\mathcal{G}(\eta, C) &:= \mathcal{G}_{\mathcal{L},(D,\mathbf{r})}(\eta, C)\\ &:= \left\{(\theta, \mu) \;\middle|\; \begin{array}{l}\theta \text{ is a metric on } K \text{ and } \eta\text{-quasisymmetric to } R_{\mathcal{E}}, \mu \in \mathcal{P}(K),\\ C^{-1} \leq r_w \mu(K_w)/(\mathrm{diam}_\theta(K_w))^2 \leq C \text{ for any } w \in W_*\end{array}\right\},\end{aligned} \tag{6.31}$$

which is considered as a subset of $C(K \times K) \times \mathcal{P}(K)$. We also set

$$\mathcal{G} := \mathcal{G}_{\mathcal{L},(D,\mathbf{r})} := \bigcup_{(\eta,C)\in \mathrm{Homeo}^+ \times (1,\infty)} \mathcal{G}(\eta, C) \tag{6.32}$$

and for each subset $\mathcal{Z}$ of $\mathcal{G}$ define $\mathcal{H}_0(\mathcal{Z}) \subset \mathcal{H}_0$ and $\widetilde{\mathcal{H}}_0(\mathcal{Z}) \subset \mathcal{H}_0/\mathbb{R}\mathbb{1}_K$ by

$$\mathcal{H}_0(\mathcal{Z}) := \{h \in \mathcal{H}_0 \mid (\theta_h, \Gamma(h, h)) \in \mathcal{Z} \text{ for some metric } \theta_h \text{ on } K\}, \tag{6.33}$$

$$\widetilde{\mathcal{H}}_0(\mathcal{Z}) := \{h + \mathbb{R}\mathbb{1}_K \mid h \in \mathcal{H}_0(\mathcal{Z})\}. \tag{6.34}$$

Since $\mu \in \mathcal{A}(K, R_{\mathcal{E}}, m, \mathcal{E}, \mathcal{F})$ for any $(\theta, \mu) \in \mathcal{G}$ by (6.12) with m in place of μ and (6.31), it follows from Proposition 6.10 with $\beta = 2$ and (1.3) that

$$\mathcal{G}(K, R_{\mathcal{E}}, m, \mathcal{E}, \mathcal{F}) = \{a\mu \mid (\theta, \mu) \in \mathcal{G}, a \in (0, \infty)\}. \tag{6.35}$$

In particular,

$\mathcal{G}(K, R_{\mathcal{E}}, m, \mathcal{E}, \mathcal{F}) \neq \emptyset$, i.e., the infimum in (1.4) is attained for $(K, R_{\mathcal{E}}, m, \mathcal{E}, \mathcal{F})$, if and only if $\mathcal{G} \neq \emptyset$, namely $\mathcal{G}(\eta, C) \neq \emptyset$ for some $(\eta, C) \in \mathrm{Homeo}^+ \times (1, \infty)$. (6.36)

In fact, it turns out that in this case $\mathcal{H}_0(\mathcal{G}) \neq \emptyset$, i.e., $(\theta_h, \Gamma(h, h)) \in \mathcal{G}$ for some $h \in \mathcal{H}_0$ and some $\theta_h \in \mathcal{J}(K, R_{\mathcal{E}})$, which is the main result of this subsection and stated as follows. *We let $c_{R_{\mathcal{E}}}$ be as in Lemma 6.11, take arbitrary $(\eta, C) \in \mathrm{Homeo}^+ \times (1, \infty)$, define $\tilde{\eta} \in \mathrm{Homeo}^+$ by $\tilde{\eta}(t) := 1/\eta^{-1}(t^{-1})$ $(\tilde{\eta}(0) := 0)$ and fix them throughout the rest of this subsection.*

Theorem 6.16 *If $\mathcal{G}(\eta, C) \neq \emptyset$, then $\mathcal{H}_0(\mathcal{G}(c_{R_{\mathcal{E}}}^{-1}\eta, C)) \neq \emptyset$, i.e., there exist $h \in \mathcal{H}_0$ and a metric θ_h on K such that $(\theta_h, \Gamma(h, h)) \in \mathcal{G}(c_{R_{\mathcal{E}}}^{-1}\eta, C)$.*

Moreover, a slight addition to our proof of Theorem 6.16 also shows the following proposition, which is used in Sect. 6.3.2 to prove the non-attainment of the infimum in (1.4) for the N-dimensional Sierpiński gasket with $N \geq 3$ (Theorem 6.35). Recall (RF2) for $(\mathcal{F}/\mathbb{R}\mathbb{1}_K, \mathcal{E})$.

Proposition 6.17 *(1) $\widetilde{\mathcal{H}}_0(\mathcal{G}(\eta, C))$ is compact in norm in $(\mathcal{F}/\mathbb{R}\mathbb{1}_K, \mathcal{E})$.*
(2) If $h \in \mathcal{H}_0(\mathcal{G}(\eta, C))$, then $\mathcal{E}(h \circ F_w, h \circ F_w)^{-1/2} h \circ F_w \in \mathcal{H}_0(\mathcal{G}(c_{R_\mathcal{E}}^{-1}\eta, C))$ for any $w \in W_*$.

We remark that in Proposition 6.17-(2) we have $\mathcal{E}(h \circ F_w, h \circ F_w) = r_w \Gamma(h, h)(K_w) > 0$ for any $w \in W_*$ by Lemma 6.21 below and the lower inequality in (6.31) for $\mu = \Gamma(h, h)$.

The rest of this subsection is devoted to the proof of Theorem 6.16 and Proposition 6.17, which is reduced to proving a series of propositions and lemmas concerning the set $\mathcal{G}(\eta, C)$. We start with establishing its compactness. Note that $\mathcal{P}(K)$ is a compact metrizable topological space by [87, Theorems 9.1.5 and 9.1.9] and hence that $C(K \times K) \times \mathcal{P}(K)$ is also metrizable.

Proposition 6.18 *$\mathcal{G}(\eta, C)$ is a compact subset of $C(K \times K) \times \mathcal{P}(K)$.*

Proof Let $\{(\theta_n, \mu_n)\}_{n\in\mathbb{N}} \subset \mathcal{G}(\eta, C)$. By the metrizability of $C(K\times K)\times\mathcal{P}(K)$ noted above, it suffices to show that there exists a subsequence of $\{(\theta_n, \mu_n)\}_{n\in\mathbb{N}}$ converging to some $(\theta, \mu) \in \mathcal{G}(\eta, C)$ in $C(K \times K) \times \mathcal{P}(K)$.

First, recalling that the compactness of K implies that of $\mathcal{P}(K)$ by [87, Theorem 9.1.9], we can choose $\mu \in \mathcal{P}(K)$ and a subsequence $\{\mu_{n_k}\}_{k\in\mathbb{N}}$ of $\{\mu_n\}_{n\in\mathbb{N}}$ converging to μ in $\mathcal{P}(K)$, and therefore by considering $\{(\theta_{n_k}, \mu_{n_k})\}_{k\in\mathbb{N}}$ instead of $\{(\theta_n, \mu_n)\}_{n\in\mathbb{N}}$ we may assume that $\{\mu_n\}_{n\in\mathbb{N}}$ itself converges to μ in $\mathcal{P}(K)$.

Next, $\mathrm{diam}_{\theta_n}(K) \in [C^{-1/2}, C^{1/2}]$ for any $n \in \mathbb{N}$ by the inequalities in (6.31) and hence $\{\theta_n\}_{n\in\mathbb{N}}$ is uniformly bounded. Moreover, for each $n \in \mathbb{N}$, since the η-quasisymmetry of θ_n to $R_\mathcal{E}$ yields the $\tilde{\eta}$-quasisymmetry of $R_\mathcal{E}$ to θ_n by [40, Proposition 10.6], it follows from (3.7) that for any $x, y \in K$,

$$\frac{R_\mathcal{E}(x, y)}{\mathrm{diam}_{R_\mathcal{E}}(K)} = \frac{\mathrm{diam}_{R_\mathcal{E}}(\{x, y\})}{\mathrm{diam}_{R_\mathcal{E}}(K)} \le \eta\Bigl(\frac{2\,\mathrm{diam}_{\theta_n}(\{x, y\})}{\mathrm{diam}_{\theta_n}(K)}\Bigr) \le \eta\bigl(2C^{1/2}\theta_n(x, y)\bigr), \tag{6.37}$$

$$C^{-1/2}\theta_n(x, y) \le \frac{\mathrm{diam}_{\theta_n}(\{x, y\})}{\mathrm{diam}_{\theta_n}(K)} \le \tilde{\eta}\Bigl(\frac{2\,\mathrm{diam}_{R_\mathcal{E}}(\{x, y\})}{\mathrm{diam}_{R_\mathcal{E}}(K)}\Bigr) = \tilde{\eta}\Bigl(\frac{2R_\mathcal{E}(x, y)}{\mathrm{diam}_{R_\mathcal{E}}(K)}\Bigr), \tag{6.38}$$

which in turn implies that for any $x_1, y_1, x_2, y_2 \in K$,

$$\begin{aligned} |\theta_n(x_1, y_1) - \theta_n(x_2, y_2)| &\le \theta_n(x_1, x_2) + \theta_n(y_1, y_2) \\ &\le C^{1/2}\biggl(\tilde{\eta}\Bigl(\frac{2R_\mathcal{E}(x_1, x_2)}{\mathrm{diam}_{R_\mathcal{E}}(K)}\Bigr) + \tilde{\eta}\Bigl(\frac{2R_\mathcal{E}(y_1, y_2)}{\mathrm{diam}_{R_\mathcal{E}}(K)}\Bigr)\biggr), \end{aligned} \tag{6.39}$$

so that $\{\theta_n\}_{n\in\mathbb{N}} \subset C(K \times K)$ is equicontinuous. Thus by the Arzelà–Ascoli theorem (see, e.g., [81, Theorem 11.28]) there exist $\theta \in C(K \times K)$ and a subsequence $\{\theta_{n_k}\}_{k\in\mathbb{N}}$ of $\{\theta_n\}_{n\in\mathbb{N}}$ converging to θ in $C(K\times K)$, so that $\{(\theta_{n_k},\mu_{n_k})\}_{k\in\mathbb{N}}$ converges to (θ,μ) in $C(K\times K)\times\mathcal{P}(K)$. Then $\mathrm{diam}_\theta(K)\in[C^{-1/2},C^{1/2}]$ and for any $x,y,z\in K$ we have (6.37) with θ in place of θ_n, $\theta(x,x)=0$, $\theta(x,y)=\theta(y,x)\geq 0$ and $\theta(x,y)\leq\theta(x,z)+\theta(z,y)$ by the same properties of θ_{n_k} for $k\in\mathbb{N}$, whence θ is a metric on K. Furthermore letting $k\to\infty$ in the η-quasisymmetry of θ_{n_k} to $R_{\mathcal{E}}$ as defined in (3.2) yields that of θ to $R_{\mathcal{E}}$.

To show the inequalities in (6.31) for (θ,μ), let $w\in W_*$, choose $x=x_w\in K_w\setminus F_w(V_0)$ and set $s:=r_w$, so that $w\in\Lambda_s$ and $\Lambda_{s,x}=\{w\}$ by [60, Proposition 1.3.5-(2)] (recall Definition 6.13). Note that $K_w=F_w(K)$ is compact and hence closed in K, that $K_w\setminus F_w(V_0)=K\setminus\bigl(F_w(V_0)\cup\bigcup_{v\in W_{|w|}\setminus\{w\}}K_v\bigr)$ by [60, Proposition 1.3.5-(2)] and is thus open in K, and that $K_w\subset U_s^\circ(x)$ with $U_s^\circ(x)$ the interior of $U_s(x)$ in K by [60, Proposition 1.3.6]. By using these facts and the convergence of $\{(\theta_{n_k},\mu_{n_k})\}_{k\in\mathbb{N}}$ to (θ,μ) in $C(K\times K)\times\mathcal{P}(K)$ to let $k\to\infty$ in the inequalities in (6.31) for (θ_{n_k},μ_{n_k}), we obtain

$$r_w\mu(K_w)\geq\limsup_{k\to\infty}r_w\mu_{n_k}(K_w)\geq C^{-1}(\mathrm{diam}_\theta(K_w))^2, \tag{6.40}$$

$$r_w\mu(K_w\setminus F_w(V_0))\leq\liminf_{k\to\infty}r_w\mu_{n_k}(K_w\setminus F_w(V_0))\leq C(\mathrm{diam}_\theta(K_w))^2, \tag{6.41}$$

$$\begin{aligned}r_w\mu(K_w)&\leq r_w\mu(U_s^\circ(x))\leq\liminf_{k\to\infty}r_w\mu_{n_k}(U_s^\circ(x))\\&\leq\sum_{v\in\Lambda_{s,x}^1}C\frac{r_w}{r_v}(\mathrm{diam}_\theta(K_v))^2\\&\leq\frac{C}{\min_{k\in S}r_k}\sum_{v\in\Lambda_{s,x}^1}(\mathrm{diam}_\theta(K_v))^2\asymp(\mathrm{diam}_\theta(K_w))^2,\end{aligned} \tag{6.42}$$

where the last step in (6.42) follows from (6.28) and (6.21). We now conclude from (6.40), (6.42) and (6.27) that $\mu(K_{wj})\asymp r_{wj}^{-1}(\mathrm{diam}_\theta(K_{wj}))^2\asymp r_w^{-1}(\mathrm{diam}_\theta(K_w))^2\asymp\mu(K_w)$ for any $(w,j)\in W_*\times S$, which together with [63, Theorem 1.2.4] implies that $\mu(F_w(V_0))=0$ and hence $\mu(K_w\setminus F_w(V_0))=\mu(K_w)$ for any $w\in W_*$, so that (6.40) and (6.41) yield the inequalities in (6.31) for (θ,μ) and thus $(\theta,\mu)\in\mathcal{G}(\eta,C)$. □

Corollary 6.19 *Let $\{(\theta_n,\mu_n)\}_{n\in\mathbb{N}}\subset\mathcal{G}(\eta,C)$, $\mu\in\mathcal{P}(K)$ and suppose that $\{\mu_n\}_{n\in\mathbb{N}}$ converges to μ in $\mathcal{P}(K)$. Then there exists a metric θ on K such that $(\theta,\mu)\in\mathcal{G}(\eta,C)$.*

Proof Since $\mathcal{G}(\eta,C)$ is a compact subset of $C(K\times K)\times\mathcal{P}(K)$ by Proposition 6.18, there exist $(\theta,\nu)\in\mathcal{G}(\eta,C)$ and a subsequence $\{(\theta_{n_k},\mu_{n_k})\}_{k\in\mathbb{N}}$ of

$\{(\theta_n, \mu_n)\}_{n\in\mathbb{N}}$ converging to (θ, ν) in $C(K \times K) \times \mathcal{P}(K)$, but then $\{\mu_{n_k}\}_{k\in\mathbb{N}}$ converges in $\mathcal{P}(K)$ to both μ and ν, hence $\mu = \nu$ and thus $(\theta, \mu) = (\theta, \nu) \in \mathcal{G}(\eta, C)$. □

We next observe that the set $\mathcal{G}(\eta, C)$ is almost invariant under the operation of pulling back by F_w followed by a suitable normalization, as stated in the following lemma.

Lemma 6.20 *Let $(\theta, \mu) \in \mathcal{G}(\eta, C)$, $w \in W_*$ and define $(\theta_w, \mu_w) \in C(K \times K) \times \mathcal{P}(K)$ by*

$$\theta_w(x, y) := \frac{\theta(F_w(x), F_w(y))}{\sqrt{r_w \mu(K_w)}} \quad \text{and} \quad \mu_w(A) := \frac{\mu(F_w(A))}{\mu(K_w)}. \tag{6.43}$$

Then $(\theta_w, \mu_w) \in \mathcal{G}(c_{R_\mathcal{E}}^{-1}\eta, C)$.

Proof It is immediate from $(\theta, \mu) \in \mathcal{G}(\eta, C)$ that θ_w and μ_w can be defined by (6.43) and are a metric and a Borel probability measure on K, respectively, and that (θ_w, μ_w) satisfies the inequalities in (6.31). Moreover, for any $x, y, z \in K$ and $t \in (0, \infty)$ with $\theta_w(x, y) \le t\theta_w(x, z)$, we have $\theta(F_w(x), F_w(y)) \le t\theta(F_w(x), F_w(z))$ and hence it follows from the η-quasisymmetry of θ to $R_\mathcal{E}$ and Lemma 6.11 that

$$c_{R_\mathcal{E}} r_w R_\mathcal{E}(x, y) \le R_\mathcal{E}(F_w(x), F_w(y)) \le \eta(t) R_\mathcal{E}(F_w(x), F_w(z)) \le \eta(t) r_w R_\mathcal{E}(x, z)$$

and thus that $R_\mathcal{E}(x, y) \le c_{R_\mathcal{E}}^{-1}\eta(t) R_\mathcal{E}(x, z)$, proving the $c_{R_\mathcal{E}}^{-1}\eta$-quasisymmetry of θ_w to $R_\mathcal{E}$. □

The operation as in (6.43) of pulling back Borel measures on K by F_w is compatible with the analogous operation on $\mathcal{F}$ (recall (6.7) and (6.8)) in the following sense.

Lemma 6.21 *Let $u \in \mathcal{F}$ and $w \in W_*$. Then $\Gamma(u, u)(F_w(A)) = r_w^{-1}\Gamma(u \circ F_w, u \circ F_w)(A)$ for any Borel subset A of K, and in particular $\Gamma(u, u)(K_w) = r_w^{-1}\mathcal{E}(u \circ F_w, u \circ F_w)$. Moreover, if $\Gamma(u, u)(K_w) > 0$, then for any Borel subset A of K,*

$$\frac{\Gamma(u, u)(F_w(A))}{\Gamma(u, u)(K_w)} = \Gamma(u_w, u_w)(A),$$
$$\text{where} \quad u_w := \mathcal{E}(u \circ F_w, u \circ F_w)^{-1/2} u \circ F_w. \tag{6.44}$$

Proof Since $F_w \colon K \to K_w$ is a homeomorphism, the first assertion is easily seen to be equivalent to [49, Lemma 4-(i)], and the second follows by choosing $A = K$ in the first. Furthermore if $\Gamma(u, u)(K_w) > 0$, then we see from the first and second assertions and the bilinearity of $\Gamma(f, g)$ in $f, g \in \mathcal{F}$ that for any Borel subset A of K,

$$\frac{\Gamma(u,u)(F_w(A))}{\Gamma(u,u)(K_w)} = \frac{r_w^{-1}\Gamma(u \circ F_w, u \circ F_w)(A)}{r_w^{-1}\mathcal{E}(u \circ F_w, u \circ F_w)} = \Gamma(u_w, u_w)(A),$$

completing the proof. $\square$

At this stage, we can already give the proof of Proposition 6.17 as follows.

Proof of Proposition 6.17 Recall (6.33) for $\mathcal{H}_0(\mathcal{Z})$ and (6.34) for $\widetilde{\mathcal{H}}_0(\mathcal{Z})$.

(2) Let $h \in \mathcal{H}_0(\mathcal{G}(\eta, C))$ and set $\mu := \Gamma(h, h)$, so that $(\theta, \mu) \in \mathcal{G}(\eta, C)$ for some metric θ on K. Let $w \in W_*$, set $h_w := \mathcal{E}(h \circ F_w, h \circ F_w)^{-1/2} h \circ F_w$ and define $(\theta_w, \mu_w) \in C(K \times K) \times \mathcal{P}(K)$ by (6.43). Then $h_w \in \mathcal{H}_0$ by Proposition 6.6-(2), $\Gamma(h_w, h_w) = \mu_w$ by Lemma 6.21, $(\theta_w, \mu_w) \in \mathcal{G}(c_{R_\mathcal{E}}^{-1}\eta, C)$ by Lemma 6.20 and thus $h_w \in \mathcal{H}_0(\mathcal{G}(c_{R_\mathcal{E}}^{-1}\eta, C))$.

(1) Let $\{h_n\}_{n\in\mathbb{N}} \subset \mathcal{H}_0(\mathcal{G}(\eta, C))$, so that $\{\Gamma(h_n, h_n)\}_{n\in\mathbb{N}} \subset \mathcal{P}(K)$ and hence $\{h_n\}_{n\in\mathbb{N}} \subset \{h \in \mathcal{H}_0 \mid (\Gamma(h, h)(K) =) \, \mathcal{E}(h, h) = 1\}$. Since $\mathcal{H}_0/\mathbb{R}\mathbb{1}_K$ is a finite-dimensional linear subspace of $(\mathcal{F}/\mathbb{R}\mathbb{1}_K, \mathcal{E})$ by Proposition 6.6-(1), $\{h \in \mathcal{H}_0/\mathbb{R}\mathbb{1}_K \mid \mathcal{E}(h, h) = 1\}$ is compact in norm in $(\mathcal{F}/\mathbb{R}\mathbb{1}_K, \mathcal{E})$, and thus there exist $h \in \mathcal{H}_0$ and a strictly increasing sequence $\{n_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ such that $\mathcal{E}(h, h) = 1$ and $\lim_{k\to\infty} \mathcal{E}(h - h_{n_k}, h - h_{n_k}) = 0$. Then $\Gamma(h, h) \in \mathcal{P}(K)$ and, noting that $\mathcal{F} \times \mathcal{F} \ni (u, v) \mapsto \Gamma(u, v)$ is bilinear, symmetric and non-negative definite, for any Borel subset A of K we have

$$\begin{aligned}\left|\Gamma(h,h)(A)^{1/2} - \Gamma(h_{n_k}, h_{n_k})(A)^{1/2}\right|^2 &\le \Gamma(h - h_{n_k}, h - h_{n_k})(A) \\ &\le \Gamma(h - h_{n_k}, h - h_{n_k})(K) = \mathcal{E}(h - h_{n_k}, h - h_{n_k}) \xrightarrow{k\to\infty} 0.\end{aligned} \tag{6.45}$$

In particular, $\{\Gamma(h_{n_k}, h_{n_k})\}_{k\in\mathbb{N}}$ converges to $\Gamma(h, h)$ in $\mathcal{P}(K)$, and now it follows from $\{h_{n_k}\}_{k\in\mathbb{N}} \subset \mathcal{H}_0(\mathcal{G}(\eta, C))$ and Corollary 6.19 that $h \in \mathcal{H}_0(\mathcal{G}(\eta, C))$, which together with $\lim_{k\to\infty} \mathcal{E}(h - h_{n_k}, h - h_{n_k}) = 0$ proves that $\widetilde{\mathcal{H}}_0(\mathcal{G}(\eta, C))$ is (sequentially) compact in norm in $(\mathcal{F}/\mathbb{R}\mathbb{1}_K, \mathcal{E})$. $\square$

We continue with the preparation for the proof of Theorem 6.16. Recall that μ is a minimal energy-dominant measure of $(\mathcal{E}, \mathcal{F})$ for any $(\theta, \mu) \in \mathcal{G}$ by Proposition 6.10 and Proposition 2.11-(b), and hence in particular that $\Gamma(u, u)$ is absolutely continuous with respect to μ for any $(\theta, \mu) \in \mathcal{G}$ and any $u \in \mathcal{F}$. The following lemma is a special case of the well-known Lebesgue differentiation theorem.

Lemma 6.22 *Let $(\theta, \mu) \in \mathcal{G}$, $u \in \mathcal{F}$ and set $f := d\Gamma(u,u)/d\mu$. Then μ-a.e. $x \in K$ is an $(R_{\mathcal{E}}, \mu)$-*Lebesgue point *for f, i.e., satisfies*

$$\lim_{s \downarrow 0} \frac{1}{\mu(B_{R_{\mathcal{E}}}(x,s))} \int_{B_{R_{\mathcal{E}}}(x,s)} |f(y) - f(x)| \, d\mu(y) = 0. \tag{6.46}$$

Proof We have $R_{\mathcal{E}} \in \mathcal{J}(K, \theta)$ by $(\theta, \mu) \in \mathcal{G}$ and hence $\theta \in \mathcal{J}(K, R_{\mathcal{E}})$ by (1.3). Thus (3.3) holds with $(d_1, d_2) = (R_{\mathcal{E}}, \theta)$, and (K, θ, μ) is VD by Proposition 6.10 and Theorem 4.5, which together imply that $(K, R_{\mathcal{E}}, \mu)$ is VD. Now since $C(K)$ is dense in $L^1(K, \mu)$ (see, e.g., [81, Theorem 3.14]) and $f \in L^1(K, \mu)$, the claim follows by Lebesgue's differentiation theorem [40, (2.8)] for $(K, R_{\mathcal{E}}, \mu)$, which requires $(K, R_{\mathcal{E}}, \mu)$ to be VD. □

Lemma 6.23 *Let $(\theta, \mu) \in \mathcal{G}$, $u \in \mathcal{F}$, let $f : K \to [0, \infty)$ be a Borel measurable μ-version of $d\Gamma(u,u)/d\mu$ and let $x \in K$ satisfy (6.46). Then for any $\omega \in \pi^{-1}(x)$ and any $w \in W_*$,*

$$\lim_{n \to \infty} \frac{\Gamma(u,u)(K_{[\omega]_n w})}{\mu(K_{[\omega]_n w})} = f(x). \tag{6.47}$$

Proof Let $\omega \in \pi^{-1}(x)$, $w \in W_*$, $n \in \mathbb{N} \cup \{0\}$ and set $s_n := \operatorname{diam}_{R_{\mathcal{E}}}(K_{[\omega]_n})$. Then by (6.22), (6.26), and VD of $(K, R_{\mathcal{E}}, \mu)$ noted in the above proof of Lemma 6.22, we have

$$\mu(K_{[\omega]_n w}) \geq c^{|w|} \mu(K_{[\omega]_n}) \geq c^{|w|} c' \mu(B_{R_{\mathcal{E}}}(x, 2s_n)) \tag{6.48}$$

for some $c, c' \in (0, \infty)$ determined solely by $\mathcal{L}$, $(D, \mathbf{r})$, (θ, μ). Now since $K_{[\omega]_n w} \subset K_{[\omega]_n} \subset B_{R_{\mathcal{E}}}(x, 2s_n)$ and $\lim_{n \to \infty} s_n = 0$ by Lemma 6.11, it follows from (6.48) and (6.46) that

$$\begin{aligned} \left| \frac{\Gamma(u,u)(K_{[\omega]_n w})}{\mu(K_{[\omega]_n w})} - f(x) \right| &= \left| \frac{1}{\mu(K_{[\omega]_n w})} \int_{K_{[\omega]_n w}} (f(y) - f(x)) \, d\mu(y) \right| \\ &\leq \frac{1}{\mu(K_{[\omega]_n w})} \int_{K_{[\omega]_n w}} |f(y) - f(x)| \, d\mu(y) \\ &\leq \frac{(c^{|w|} c')^{-1}}{\mu(B_{R_{\mathcal{E}}}(x, 2s_n))} \int_{B_{R_{\mathcal{E}}}(x, 2s_n)} |f(y) - f(x)| \, d\mu(y) \\ &\xrightarrow{m \to \infty} 0, \end{aligned}$$

proving (6.47). □

Taking an $(R_{\mathcal{E}}, \mu)$-Lebesgue point $x \in K$ for $d\Gamma(u,u)/d\mu$ with $(d\Gamma(u,u)/d\mu)(x) > 0$ and considering the enlargements of infinitesimally small cells containing x to the original scale as in Lemmas 6.20 and 6.21, we arrive at the following proposition.

Proposition 6.24 *Let* $(\theta, \mu) \in \mathcal{G}$, $u \in \mathcal{F}$, *let* $f \colon K \to [0, \infty)$ *be a Borel measurable* μ*-version of* $d\Gamma(u,u)/d\mu$, *let* $x \in K$ *satisfy* (6.46) *and* $f(x) > 0$, *and let* $\omega \in \pi^{-1}(x)$. *For each* $n \in \mathbb{N} \cup \{0\}$, *define* $\mu_n := \mu_{[\omega]_n} \in \mathcal{P}(K)$ *by* (6.43) *with* $w = [\omega]_n$ *and, noting that* $\Gamma(u,u)(K_{[\omega]_n}) > 0$ *by* (6.47), *define* $u_n := u_{[\omega]_n} \in \mathcal{F}$ *by* (6.44) *with* $w = [\omega]_n$. *If* $v \in \mathcal{F}$ *and* $\{n_k\}_{k \in \mathbb{N}} \subset \mathbb{N}$ *is strictly increasing and satisfies* $\lim_{k\to\infty} \mathcal{E}(v - u_{n_k}, v - u_{n_k}) = 0$, *then* $\Gamma(v,v) \in \mathcal{P}(K)$ *and* $\{\mu_{n_k}\}_{k\in\mathbb{N}}$ *converges to* $\Gamma(v,v)$ *in* $\mathcal{P}(K)$.

Proof Let $w \in W_*$. Then we see from (6.43), (6.44), (6.47) and $f(x) \in (0, \infty)$ that

$$\frac{\Gamma(u_n,u_n)(K_w)}{\mu_n(K_w)} = \frac{\Gamma(u,u)(K_{[\omega]_n w})/\mu(K_{[\omega]_n w})}{\Gamma(u,u)(K_{[\omega]_n})/\mu(K_{[\omega]_n})} \xrightarrow{n\to\infty} \frac{f(x)}{f(x)} = 1. \tag{6.49}$$

On the other hand, the same argument as (6.45) based on $\lim_{k\to\infty} \mathcal{E}(v - u_{n_k}, v - u_{n_k}) = 0$ yields $\lim_{n\to\infty} \Gamma(u_{n_k}, u_{n_k})(K_w) = \Gamma(v,v)(K_w)$, which together with (6.49) implies that

$$\lim_{k\to\infty} \mu_{n_k}(K_w) = \Gamma(v,v)(K_w) \tag{6.50}$$

and in particular that $\Gamma(v,v)(K) = \lim_{k\to\infty} \mu_{n_k}(K) = 1$, namely $\Gamma(v,v) \in \mathcal{P}(K)$. Note that $\mu_n(F_w(V_0)) = \mu(K_{[\omega]_n})^{-1}\mu(F_{[\omega]_n w}(V_0)) = 0$ for any $n \in \mathbb{N} \cup \{0\}$ by Proposition 6.10 and that $\Gamma(v,v)(F_w(V_0)) = 0$ by $\#F_w(V_0) < \infty$ and [24, Theorem 4.3.8], and recall that $K_w = F_w(K)$ is closed in K and $K_w \setminus F_w(V_0)$ is open in K as noted in the last paragraph of the proof of Proposition 6.18. By using these facts and the equality $K \setminus V_n = \bigcup_{v \in W_n} (K_v \setminus F_v(V_0))$, with the union disjoint, implied by [60, Proposition 1.3.5-(2)] for any $n \in \mathbb{N}$, we easily see that the validity of (6.50) for any $w \in W_*$ is equivalent to the desired convergence of $\{\mu_{n_k}\}_{k\in\mathbb{N}}$ to $\Gamma(v,v)$ in $\mathcal{P}(K)$, which completes the proof. □

Now we can conclude the proof of the main result of this subsection (Theorem 6.16).

Proof of Theorem 6.16 By the assumption $\mathcal{G}(\eta, C) \neq \emptyset$ we can take $(\theta, \mu) \in \mathcal{G}(\eta, C)$. Let $u \in \mathcal{H}_0 \setminus \mathbb{R}\mathbb{1}_K$, which exists by Proposition 6.6-(1) and $\#V_0 \geq 2$, and let $f \colon K \to [0, \infty)$ be a Borel measurable μ-version of $d\Gamma(u,u)/d\mu$. Then since $\mu\bigl(f^{-1}((0,\infty))\bigr) > 0$ by $\int_K f\, d\mu = \Gamma(u,u)(K) = \mathcal{E}(u,u) > 0$, Lemma 6.22 implies that there exists $x \in K$ with the properties (6.46) and $f(x) > 0$. Let $\omega \in \pi^{-1}(x)$, and for each $n \in \mathbb{N} \cup \{0\}$, as in Proposition 6.24

define $(\theta_n, \mu_n) := (\theta_{[\omega]_n}, \mu_{[\omega]_n}) \in C(K\times K)\times\mathcal{P}(K)$ by (6.43) with $w = [\omega]_n$ and $u_n := u_{[\omega]_n} \in \mathcal{F}$ by (6.44) with $w = [\omega]_n$, so that $\{(\theta_n, \mu_n)\}_{n\in\mathbb{N}\cup\{0\}} \subset \mathcal{G}(c_{R_{\mathcal{E}}}^{-1}\eta, C)$ by Lemma 6.20 and $\{u_n\}_{n\in\mathbb{N}\cup\{0\}} \subset \{h \in \mathcal{H}_0 \mid \mathcal{E}(h, h) = 1\}$ by Proposition 6.6-(2). Noting that $\mathcal{H}_0/\mathbb{R}\mathbb{1}_K$ is a finite-dimensional linear subspace of the Hilbert space $(\mathcal{F}/\mathbb{R}\mathbb{1}_K, \mathcal{E})$ by Proposition 6.6-(1) and hence that $\{h \in \mathcal{H}_0/\mathbb{R}\mathbb{1}_K \mid \mathcal{E}(h, h) = 1\}$ is compact in norm in $(\mathcal{F}/\mathbb{R}\mathbb{1}_K, \mathcal{E})$, for some $h \in \mathcal{H}_0$ and some strictly increasing sequence $\{n_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ we have $\mathcal{E}(h, h) = 1$ and $\lim_{k\to\infty} \mathcal{E}(h - u_{n_k}, h - u_{n_k}) = 0$. Then $\Gamma(h, h) \in \mathcal{P}(K)$ and $\{\mu_{n_k}\}_{k\in\mathbb{N}}$ converges to $\Gamma(h, h)$ in $\mathcal{P}(K)$ by Proposition 6.24, and it follows from this convergence, $\{(\theta_{n_k}, \mu_{n_k})\}_{k\in\mathbb{N}} \subset \mathcal{G}(c_{R_{\mathcal{E}}}^{-1}\eta, C)$ and Corollary 6.19 that $(\theta_h, \Gamma(h, h)) \in \mathcal{G}(c_{R_{\mathcal{E}}}^{-1}\eta, C)$ for some metric θ_h on K. □

6.3 Examples

In this subsection, we show that the infimum in (1.4) defining the conformal walk dimension $d_{\mathrm{cw}} = 2$ *fails* to be attained for some concrete examples of post-critically finite self-similar sets. In view of (6.36) and Theorem 6.16, for the proof of the non-attainment $\mathcal{G}(K, R_{\mathcal{E}}, m, \mathcal{E}, \mathcal{F}) = \emptyset$ it suffices to verify that the conclusion of Theorem 6.16 cannot hold for any $h \in \mathcal{H}_0$. We start with providing a further characterization of $h \in \mathcal{H}_0$ as in the conclusion of Theorem 6.16. In the following definition and proposition, we continue to assume the setting specified in the first paragraph of Sect. 6.2.

Definition 6.25 Let μ be a Borel measure on K. Recalling Definition 2.3, we define the *μ-intrinsic metric* $d_{\mathrm{int}}^{\mu}\colon K \times K \to [0, \infty]$ of $(\mathcal{E}, \mathcal{F})$ by

$$d_{\mathrm{int}}^{\mu}(x, y) := \sup\{u(x) - u(y) \mid u \in \mathcal{F}, \Gamma(u, u) \le \mu\}. \tag{6.51}$$

Proposition 6.26 *(1) There exists $C \in (1, \infty)$ such that for any $h \in \mathcal{H}_0$ and any $w \in W_*$,*

$$C^{-1}\bigl(\mathrm{diam}(K_w, d_{\mathrm{int}}^{\Gamma(h,h)})\bigr)^2 \le r_w\Gamma(h, h)(K_w) \le C\bigl(\mathrm{diam}(K_w, d_{\mathrm{int}}^{\Gamma(h,h)})\bigr)^2. \tag{6.52}$$

(2) Let $h \in \mathcal{H}_0$. Then $\Gamma(h, h) \in \mathcal{G}(K, R_{\mathcal{E}}, m, \mathcal{E}, \mathcal{F})$ if and only if $d_{\mathrm{int}}^{\Gamma(h,h)} \in \mathcal{J}(K, R_{\mathcal{E}})$.

(3) $\mathcal{G}(K, R_{\mathcal{E}}, m, \mathcal{E}, \mathcal{F}) = \emptyset$ if and only if $d_{\mathrm{int}}^{\Gamma(h,h)} \notin \mathcal{J}(K, R_{\mathcal{E}})$ for any $h \in \mathcal{H}_0$.

Proof (1) Let $h \in \mathcal{H}_0$ and $w \in W_*$. Since $|h(x) - h(y)| \le d_{\mathrm{int}}^{\Gamma(h,h)}(x, y)$ for any $x, y \in K$ by (6.51) with $\mu = \Gamma(h, h)$, we see from Lemma 6.21,

$h \in \mathcal{H}_0$, (6.11), (6.10) and (6.2) that

$$r_w \Gamma(h,h)(K_w) = \mathcal{E}^{(0)}(h \circ F_w|_{V_0}, h \circ F_w|_{V_0}) \le C\bigl(\operatorname{diam}(K_w, d_{\mathrm{int}}^{\Gamma(h,h)})\bigr)^2,$$

where $C := \frac{1}{2}\sum_{p,q\in V_0,\, p\neq q} D_{pq}$. On the other hand, setting $C' := \operatorname{diam}_{R_{\mathcal{E}}}(K)$, for any $x, y \in K_w$ and any $u \in \mathcal{F}$ with $\Gamma(u,u) \le \Gamma(h,h)$, by (RF4) and Lemma 6.21 we have

$$\begin{aligned} |u(x) - u(y)|^2 &= \bigl|(u \circ F_w)(F_w^{-1}(x)) - (u \circ F_w)(F_w^{-1}(y))\bigr|^2 \\ &\le C'\mathcal{E}(u \circ F_w, u \circ F_w) = C' r_w \Gamma(u,u)(K_w) \\ &\le C' r_w \Gamma(h,h)(K_w), \end{aligned}$$

therefore taking the supremum over such u yields $d_{\mathrm{int}}^{\Gamma(h,h)}(x,y)^2 \le C' r_w \Gamma(h,h)(K_w)$ by (6.51) with $\mu = \Gamma(h,h)$ and thus $\bigl(\operatorname{diam}(K_w, d_{\mathrm{int}}^{\Gamma(h,h)})\bigr)^2 \le C' r_w \Gamma(h,h)(K_w)$, proving (6.52).

(2) If $\Gamma(h,h) \in \mathcal{G}(K, R_{\mathcal{E}}, m, \mathcal{E}, \mathcal{F})$ then $d_{\mathrm{int}}^{\Gamma(h,h)} \in \mathcal{J}(K, R_{\mathcal{E}})$ by Proposition 2.11-(a) (recall (5.3)), and conversely if $d_{\mathrm{int}}^{\Gamma(h,h)} \in \mathcal{J}(K, R_{\mathcal{E}})$ then $\Gamma(h,h) \in \mathcal{G}(K, R_{\mathcal{E}}, m, \mathcal{E}, \mathcal{F})$ by (6.52) and Proposition 6.10 with $\beta = 2$.

(3) This is immediate from (2) and the fact that, by (6.36), Theorem 6.16 and (6.35), $\mathcal{G}(K, R_{\mathcal{E}}, m, \mathcal{E}, \mathcal{F}) = \emptyset$ if and only if $\Gamma(h,h) \notin \mathcal{G}(K, R_{\mathcal{E}}, m, \mathcal{E}, \mathcal{F})$ for any $h \in \mathcal{H}_0$. □

6.3.1 The Vicsek set

Example 6.27 (Vicsek set) Set $S := \{0,1,2,3,4\}$, define $\{q_i\}_{i\in S} \subset \mathbb{R}^2$ by $q_0 := (0,0)$, $q_1 := (1,1)$, $q_2 := (-1,1)$, $q_3 := (-1,-1)$ and $q_4 := (1,-1)$, and define $f_i\colon \mathbb{R}^2 \to \mathbb{R}^2$ for each $i \in S$ by $f_i(x) := q_i + \frac{1}{3}(x - q_i)$. Let K be the *self-similar set* associated with $\{f_i\}_{i\in S}$, i.e., the unique non-empty compact subset of $\mathbb{R}^2$ such that $K = \bigcup_{i\in S} f_i(K)$, which exists and satisfies $K \subsetneq [-1,1]^2$ thanks to $\bigcup_{i\in S} f_i([-1,1]^2) \subsetneq [-1,1]^2$ by [60, Theorem 1.1.4], and set $F_i := f_i|_K$ for each $i \in S$. Then $\mathcal{L} := (K, S, \{F_i\}_{i\in S})$ is a self-similar structure by [60, Theorem 1.2.3] and called the *Vicsek set* (see Fig. 1 below), and it easily follows from $K \subset [-1,1]^2$ that $\mathcal{P}_{\mathcal{L}} = \{1^\infty, 2^\infty, 3^\infty, 4^\infty\}$ and $V_0 = \{q_1, q_2, q_3, q_4\}$, so that $\mathcal{L}$ is post-critically finite and K is connected. Let $d\colon K \times K \to [0,\infty)$ be the Euclidean metric on K given by $d(x,y) := |x-y|$.

Let $r \in \bigl(0, \frac{1}{2}\bigr)$, set $\mathbf{r} = (r_i)_{i\in S} := (1-2r, r, r, r, r)$, and define $D = (D_{pq})_{p,q\in V_0}$ and $D' = (D'_{pq})_{p,q\in V_0\cup\{q_0\}}$ by

$$D_{pq} := \begin{cases} 1 & \text{if } p \neq q, \\ -3 & \text{if } p = q, \end{cases} \qquad D'_{pq} := \begin{cases} 4 & \text{if } p \neq q \text{ and } q_0 \in \{p, q\}, \\ 0 & \text{if } p \neq q \text{ and } p, q \in V_0, \\ -16 & \text{if } p = q = q_0, \\ -4 & \text{if } p = q \in V_0. \end{cases}$$

Then setting $\mathcal{E}'^{(0)}(u, v) := -\sum_{p,q \in V_0 \cup \{q_0\}} D'_{pq} u(q) v(p)$ for $u, v \in \mathbb{R}^{V_0 \cup \{q_0\}}$, we immediately see that $\mathcal{E}^{(0)}(u, u) = \inf_{v \in \mathbb{R}^{V_0 \cup \{q_0\}},\, v|_{V_0} = u} \mathcal{E}'^{(0)}(v, v)$ for any $u \in \mathbb{R}^{V_0 \cup \{q_0\}}$, which in turn easily implies that $(D, \mathbf{r})$ is a regular harmonic structure on $\mathcal{L}$.

Our concern is whether $\mathcal{G}(K, R_{\mathcal{E}}, m, \mathcal{E}, \mathcal{F}) = \emptyset$ for the MMD space $(K, R_{\mathcal{E}}, m, \mathcal{E}, \mathcal{F})$ resulting from this $\mathcal{L}$, $(D, \mathbf{r})$. We first remark that the resistance metric $R_{\mathcal{E}}$ is quasisymmetric to the Euclidean metric d and that it is bi-Lipschitz equivalent to d when $r = \frac{1}{3}$.

Lemma 6.28 *Let $\mathcal{L} = (K, S, \{F_i\}_{i \in S}), d, (D, \mathbf{r} = (1 - 2r, r, r, r, r))$ be as in Example 6.27 and let $(K, R_{\mathcal{E}}, m, \mathcal{E}, \mathcal{F})$ be the MMD space resulting from $\mathcal{L}$, $(D, \mathbf{r})$ as introduced in Sect. 6.1. Then $R_{\mathcal{E}} \in \mathcal{J}(K, d)$. Moreover, $R_{\mathcal{E}}$ is bi-Lipschitz equivalent to d if $r = \frac{1}{3}$.*

Proof To see $R_{\mathcal{E}} \in \mathcal{J}(K, d)$, we verify that $R_{\mathcal{E}}$ and d satisfy the assumptions of [67, Theorem 3.6.6], i.e., that they are adapted in the sense of [67, Definition 2.4.7 and Proposition 2.4.8] and exponential in the sense of [67, Definitions 3.1.15 and 3.6.2] and that $R_{\mathcal{E}}$ is gentle with respect to d in the sense of [67, Definition 3.3.1]. Indeed, $R_{\mathcal{E}}$ is adapted by Lemma 6.14 and exponential by Lemma 6.11, d is obviously exponential, and d is adapted since Lemma 6.14 is easily seen to hold also with d, $(\frac{1}{3})_{i \in S}$ in place of $R_{\mathcal{E}}$, $\mathbf{r}$. Moreover, if $w, v \in W_*$ satisfy $w \neq v$, $|w| = |v|$ and $K_w \cap K_v \neq \emptyset$, then for some $\tau \in W_*$, $n \in \mathbb{N} \cup \{0\}$ and $i, j \in S \setminus \{0\}$ with $|i - j| = 2$ we have $\{w, v\} = \{\tau 0 i^n, \tau i j^n\}$ and hence $r_w / r_v \in \{r_0 / r_i, r_i / r_0\} = \{(1 - 2r)/r, r/(1 - 2r)\}$, which together with Lemma 6.11 shows that $R_{\mathcal{E}}$ is gentle with respect to d. Thus [67, Theorem 3.6.6] is applicable to $R_{\mathcal{E}}$ and d and yields $R_{\mathcal{E}} \in \mathcal{J}(K, d)$. Lastly, if $r = \frac{1}{3}$, then we have Lemma 6.14 with d in place of $R_{\mathcal{E}}$, which together with Lemma 6.14 implies that $R_{\mathcal{E}}$ is bi-Lipschitz equivalent to d. □

In the situation of Example 6.27, it turns out that the energy measure $\Gamma(h, h)$ of any $\mathcal{E}$-harmonic function $h \in \mathcal{H}_0$ has its support within the union of the diagonals of $[-1, 1]^2$, as stated in the following proposition.

Proposition 6.29 *Let $\mathcal{L} = (K, S, \{F_i\}_{i \in S})$ and $(D, \mathbf{r})$ be as in Example 6.27 and consider the MMD space $(K, R_{\mathcal{E}}, m, \mathcal{E}, \mathcal{F})$ resulting from $\mathcal{L}$, $(D, \mathbf{r})$ as introduced in Sect. 6.1. Set $I_{i,i+2} := \{(1 - t) q_i + t q_{i+2} \mid t \in [0, 1]\}$ for $i \in \{1, 2\}$. Then $\Gamma(h, h)\big(K \setminus (I_{1,3} \cup I_{2,4})\big) = 0$ for any $h \in \mathcal{H}_0$.*

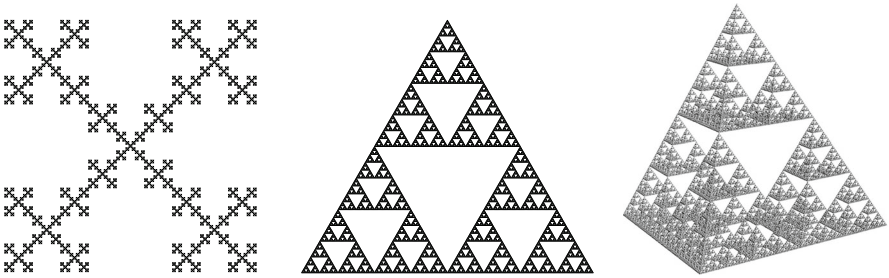

Fig. 1 The Vicsek set and the N-dimensional Sierpiński gaskets ($N = 2, 3$)

Proof Let U be a connected component of $K \setminus (I_{1,3} \cup I_{2,4})$. Then it is immediate from $K = \bigcup_{i \in S} F_i(K) \subsetneq [-1, 1]^2$ that $\partial_K U$ consists of a unique element $q_U \in I_{1,3} \cup I_{2,4}$, where $\partial_K U$ denotes the boundary of U in K. Note that $u\mathbb{1}_U, u\mathbb{1}_{K \setminus U} \in \mathcal{F}$ and $\mathcal{E}(u\mathbb{1}_U, u\mathbb{1}_{K \setminus U}) = 0$ for any $u \in \mathcal{F}$ with $u(q_U) = 0$; indeed, this is clear by [33, Exercise 1.4.1 and Theorem 1.4.2-(ii)] and the strong locality of $(\mathcal{E}, \mathcal{F})$ if $q_U \notin \mathrm{supp}_m[u]$, and the general case follows by approximating u by $\{u - (-\frac{1}{n}) \vee (u \wedge \frac{1}{n})\}_{n \in \mathbb{N}}$ on the basis of [33, Theorem 1.4.2-(iv)]. Therefore for any $u \in \mathcal{F}$, the function $u_U \in C(K)$ defined by $u_U|_U := u(q_U)\mathbb{1}_U$ and $u_U|_{K \setminus U} := u|_{K \setminus U}$ satisfies $u_U = (u - u(q_U))\mathbb{1}_{K \setminus U} + u(q_U)\mathbb{1}_K \in \mathcal{F}$ and $\mathcal{E}(u, u) \geq \mathcal{E}(u_U, u_U)$, where the equality holds if and only if $u = u_U$ by (RF1). In particular, any $u \in \mathcal{F}$ with $u \neq u_U$ fails to be 0-harmonic by $U \cap V_0 = \emptyset$, so that any $h \in \mathcal{H}_0$ satisfies $h|_U = h(q_U)\mathbb{1}_U$, hence $\Gamma(h, h)(U) = 0$ by [33, Corollary 3.2.1] and thus $\Gamma(h, h)\bigl(K \setminus (I_{1,3} \cup I_{2,4})\bigr) = 0$ since U is any one of the countably many connected components of $K \setminus (I_{1,3} \cup I_{2,4})$. $\square$

Corollary 6.30 *Let $\mathcal{L} = (K, S, \{F_i\}_{i \in S})$ and $(D, \mathbf{r})$ be as in Example 6.27 and consider the MMD space $(K, R_{\mathcal{E}}, m, \mathcal{E}, \mathcal{F})$ resulting from $\mathcal{L}$, $(D, \mathbf{r})$ as introduced in Sect. 6.1. Then $\mathcal{G}(K, R_{\mathcal{E}}, m, \mathcal{E}, \mathcal{F}) = \emptyset$, i.e., the infimum in (1.4) is not attained for $(K, R_{\mathcal{E}}, m, \mathcal{E}, \mathcal{F})$.*

Proof For any $h \in \mathcal{H}_0$ we have $\Gamma(h, h) \notin \mathcal{A}(K, R_{\mathcal{E}}, m, \mathcal{E}, \mathcal{F})$ since $\Gamma(h, h)$ does not have full support by Proposition 6.29, and therefore $\Gamma(h, h) \notin \mathcal{G}(K, R_{\mathcal{E}}, m, \mathcal{E}, \mathcal{F})$ for any $h \in \mathcal{H}_0$, which is equivalent to $\mathcal{G}(K, R_{\mathcal{E}}, m, \mathcal{E}, \mathcal{F}) = \emptyset$ by (6.36), Theorem 6.16 and (6.35) as already noted in the proof of Proposition 6.26-(3). $\square$

6.3.2 *Higher-dimensional Sierpiński gaskets*

Example 6.31 (N-dimensional Sierpiński gasket) Let $N \in \mathbb{N}$, $N \geq 2$ and let $\{q_k\}_{k=0}^N \subset \mathbb{R}^N$ be the set of the vertices of a regular N-dimensional simplex $\triangle_N$, so that $\triangle_N$ is the convex hull of $\{q_k \mid k \in \{0, 1, \ldots, N\}\}$ in $\mathbb{R}^N$. We further set $S := \{0, 1, \ldots, N\}$ and for each $i \in S$ define $f_i : \mathbb{R}^N \to \mathbb{R}^N$ by $f_i(x) := q_i + \frac{1}{2}(x - q_i)$. Let K be the self-similar set associated with $\{f_i\}_{i \in S}$, which exists and satisfies $K \subsetneqq \triangle_N$ thanks to $\bigcup_{i \in S} f_i(\triangle_N) \subsetneqq \triangle_N$ by [60, Theorem 1.1.4], and set $F_i := f_i|_K$ for each $i \in S$. Then $\mathcal{L} := (K, S, \{F_i\}_{i \in S})$ is a self-similar structure by [60, Theorem 1.2.3] and called the *N-dimensional (standard) Sierpiński gasket* (see Fig. 1 above), and it easily follows from $K \subset \triangle_N$ that $\mathcal{P}_{\mathcal{L}} = \{k^\infty \mid k \in S\}$ and $V_0 = \{q_k \mid k \in S\}$, so that $\mathcal{L}$ is post-critically finite and K is connected. Let $d : K \times K \to [0, \infty)$ be the Euclidean metric on K given by $d(x, y) := |x - y|$.

Define $D = (D_{pq})_{p,q \in V_0}$ by $D_{pp} := -N$ and $D_{pq} := 1$ for $p, q \in V_0$ with $p \neq q$. By the symmetry of $\mathcal{L}$ and D, there exists a unique $r \in (0, \infty)$ such that $(D, \mathbf{r} = (r_i)_{i \in S})$ with $r_i := r$ is a harmonic structure on $\mathcal{L}$, and an elementary calculation shows that $r = \frac{N+1}{N+3} < 1$, so that $(D, \mathbf{r})$ is a regular harmonic structure on $\mathcal{L}$.

In the rest of this subsection, we fix the setting of Example 6.31 and consider the MMD space $(K, R_{\mathcal{E}}, m, \mathcal{E}, \mathcal{F})$ resulting from $\mathcal{L}$, $(D, \mathbf{r})$ as introduced in Sect. 6.1. We first remark that the resistance metric $R_{\mathcal{E}}$ is bi-Lipschitz equivalent to the power $d^{\log_2(1/r)}$ of the Euclidean metric d and hence quasisymmetric to d.

Lemma 6.32 *$R_{\mathcal{E}}$ is bi-Lipschitz equivalent to $d^{\log_2(1/r)}$. In particular, $R_{\mathcal{E}} \in \mathcal{J}(K, d)$.*

Proof Lemma 6.14 with $d^{\log_2(1/r)}$ in place of $R_{\mathcal{E}}$ is easily seen to hold and, in combination with Lemma 6.14, immediately implies the assertions. $\square$

The following result, which is essentially due to Kigami [62], was the starting point of the whole study of the present paper.

Theorem 6.33 ([52,62]) *Assume that $N = 2$, and let $h_1, h_2 \in \mathcal{H}_0$ satisfy $\mathcal{E}(h_1, h_1) = \mathcal{E}(h_2, h_2) = 1$ and $\mathcal{E}(h_1, h_2) = 0$. Then $d_{\mathrm{int}}^{\Gamma(h_1,h_1)+\Gamma(h_2,h_2)}$, $d_{\mathrm{int}}^{\Gamma(h_1,h_1)} \in \mathcal{J}(K, R_{\mathcal{E}})$ and $\Gamma(h_1, h_1) + \Gamma(h_2, h_2), \Gamma(h_1, h_1) \in \mathcal{G}(K, R_{\mathcal{E}}, m, \mathcal{E}, \mathcal{F})$.*

Proof Set $\mu_{h_1,h_2} := \Gamma(h_1, h_1) + \Gamma(h_2, h_2)$ and let $\mu \in \{\mu_{h_1,h_2}, \Gamma(h_1, h_1)\}$. As in the proof of Lemma 6.28 above, to see $d_{\mathrm{int}}^{\mu} \in \mathcal{J}(K, R_{\mathcal{E}})$ we apply [67, Theorem 3.6.6]. Note that the results in [62, Sections 5 and 6] and [52, Sections 3 and 4] are applicable to d_{int}^{μ} by virtue of the identification of d_{int}^{μ} given in [52, Theorem 4.2]. By [62, Theorem 5.11] for $\mu = \mu_{h_1,h_2}$ and [52, Proposition 3.16-(1)] for $\mu = \Gamma(h_1, h_1)$, d_{int}^{μ} is a metric on K compatible with the original topology of K and adapted in the sense of [67, Definition 2.4.7 and Proposition 2.4.8]. Also by [62, Theorem 5.11 and Proof of Theorem 3.2] for $\mu = \mu_{h_1,h_2}$ and [52, Proposition 3.16-(1) and Lemma 3.9] for $\mu =$

$\Gamma(h_1, h_1)$, d^{μ}_{int} is exponential in the sense of [67, Definitions 3.1.15 and 3.6.2] and gentle with respect to d in the sense of [67, Definition 3.3.1]. Clearly d is exponential in the sense of [67], and is also adapted in the sense of [67] by Lemma 6.14 with $d^{\log_2(5/3)}$ in place of $R_{\mathcal{E}}$ mentioned in the above proof of Lemma 6.32. Thus [67, Theorem 3.6.6] is applicable to d^{μ}_{int} and d and shows, together with Lemma 6.32 and (1.3), that $d^{\mu}_{\mathrm{int}} \in \mathcal{J}(K, d) = \mathcal{J}(K, R_{\mathcal{E}})$. Finally, $(K, d^{\mu}_{\mathrm{int}}, \mu, \mathcal{E}, \mathcal{F})$ satisfies VD and HKE(2) by [62, Theorems 6.2 and 6.3] for $\mu = \mu_{h_1,h_2}$ and by [52, Theorem 3.19 and Corollary 4.3] for $\mu = \Gamma(h_1, h_1)$, and it thus satisfies PHI(2) by Theorem 4.5, proving $\mu \in \mathcal{G}(K, R_{\mathcal{E}}, m, \mathcal{E}, \mathcal{F})$. □

One of the key observations for the validity of Theorem 6.33 is that the energy measures of harmonic functions are volume doubling with respect to the resistance metric $R_{\mathcal{E}}$ (or equivalently, with respect to the Euclidean metric d on K), which in fact extends to the N-dimensional Sierpiński gasket with $N \geq 3$ as follows.

Proposition 6.34 $\bigl(K, R_{\mathcal{E}}, \Gamma(h, h)\bigr)$ *is* VD *for any* $h \in \mathcal{H}_0 \setminus \mathbb{R}\mathbb{1}_K$. *More generally, if* $\{h_n\}_{n\in\mathbb{N}} \subset \mathcal{H}_0$ *satisfies* $\sum_{n\in\mathbb{N}} \mathcal{E}(h_n, h_n) \in (0, \infty)$, *then* $\bigl(K, R_{\mathcal{E}}, \sum_{n\in\mathbb{N}} \Gamma(h_n, h_n)\bigr)$ *is* VD.

Proof We follow [62, Proof of Theorem 3.2]. Let $h \in \mathcal{H}_0 \setminus \mathbb{R}\mathbb{1}_K$. As noted in the second paragraph of the proof of Proposition 6.10, $\bigl(K, R_{\mathcal{E}}, \Gamma(h, h)\bigr)$ is VD if and only if (6.22) and (6.23) with $\mu = \Gamma(h, h)$ hold. To verify (6.22) and (6.23), recalling Proposition 6.6-(1) and (RF1), for each $j \in S$ we choose a basis $\{h^j_k\}_{k=0}^N$ of the linear space $\mathcal{H}_0$ such that $h^j_0 = \mathbb{1}_K$, $h^j_1|_{V_0} = N^{-1/2}\mathbb{1}_{V_0\setminus\{q_j\}}$, $h^j_k(q_j) = 0$ and $\sum_{q\in V_0} h^j_k(q) = 0$ for any $k \in \{2, \dots, N\}$ and $\{h^j_k\}_{k=2}^N$ is orthonormal in $(\mathcal{H}_0/\mathbb{R}\mathbb{1}_K, \mathcal{E})$. Then $\{h^j_k\}_{k=1}^N$ is also orthonormal in $(\mathcal{H}_0/\mathbb{R}\mathbb{1}_K, \mathcal{E})$ by (6.10), and it easily follows by $\mathcal{H}_0 \subset \mathcal{H}_1$, (6.11) and solving [60, (3.2.1)] that $h^j_1 \circ F_j = \frac{N+1}{N+3} h^j_1$ and that $h^j_k \circ F_j = \frac{1}{N+3} h^j_k$ for any $k \in \{2, \dots, N\}$. In particular, for each $w \in W_*$, a linear map $F^*_w \colon \mathcal{H}_0/\mathbb{R}\mathbb{1}_K \to \mathcal{H}_0/\mathbb{R}\mathbb{1}_K$ is defined by $F^*_w(u + \mathbb{R}\mathbb{1}_K) := u \circ F_w + \mathbb{R}\mathbb{1}_K$ and is invertible, and we set $h_w := \mathcal{E}(h \circ F_w, h \circ F_w)^{-1/2} h \circ F_w \in \mathcal{H}_0 \setminus \mathbb{R}\mathbb{1}_K$.

Let $(w, j) \in W_* \times S$. Then $h_w = \sum_{k=0}^N a_k h^j_k$ for some $(a_k)_{k=0}^N \in \mathbb{R}^{N+1}$. Since $h_w \circ F_j = a_0 h^j_0 + \frac{N+1}{N+3} a_1 h^j_1 + \frac{1}{N+3} \sum_{k=2}^N a_k h^j_k$ and $\sum_{k=1}^N a_k^2 = \mathcal{E}(h_w, h_w) = 1$, by Lemma 6.21 we have

$$\frac{\frac{N+1}{N+3}\Gamma(h, h)(K_{wj})}{\Gamma(h, h)(K_w)} = \mathcal{E}(h_w \circ F_j, h_w \circ F_j)$$
$$= \frac{(N+1)^2 a_1^2 + \sum_{k=2}^N a_k^2}{(N+3)^2} \in \left[\frac{1}{(N+3)^2}, \frac{(N+1)^2}{(N+3)^2}\right], \tag{6.53}$$

proving (6.22) with $\mu = \Gamma(h, h)$.

Next, to show (6.23), let $w, v \in W_*$ satisfy $|w| = |v|$ and $K_w \cap K_v \neq \emptyset$. We may assume $w \neq v$, so that there exist $\tau \in W_*$, $n \in \mathbb{N} \cup \{0\}$ and $i, j \in S$ with $i \neq j$ such that $w = \tau i j^n$ and $v = \tau j i^n$. Take $(b_k)_{k=0}^N, (c_k)_{k=0}^N \in \mathbb{R}^{N+1}$ such that $h_\tau \circ F_i = \sum_{k=0}^N b_k h_k^j$ and $h_\tau \circ F_j = \sum_{k=0}^N c_k h_k^i$. Then $b_0 = h_\tau(F_i(q_j)) = h_\tau(F_j(q_i)) = c_0$ by $F_i(q_j) = F_j(q_i)$, $Nb_0 + N^{1/2} b_1 + Nc_0 + N^{1/2} c_1 = 2Nb_0$ by the harmonicity of h_τ at $F_i(q_j)$ (see [60, (3.2.1)]) and thus $b_1 = -c_1$. Moreover, by (6.53),

$$\frac{\sum_{k=1}^N b_k^2}{\sum_{k=1}^N c_k^2} = \frac{\mathcal{E}(h_\tau \circ F_i, h_\tau \circ F_i)}{\mathcal{E}(h_\tau \circ F_j, h_\tau \circ F_j)} \leq \frac{(N+1)^2/(N+3)^2}{1/(N+3)^2} = (N+1)^2. \tag{6.54}$$

Since $h_\tau \circ F_{ij^n} = b_0 h_0^j + \bigl(\frac{N+1}{N+3}\bigr)^n b_1 h_1^j + \bigl(\frac{1}{N+3}\bigr)^n \sum_{k=2}^N b_k h_k^j$ and $h_\tau \circ F_{ji^n} = c_0 h_0^i + \bigl(\frac{N+1}{N+3}\bigr)^n c_1 h_1^i + \bigl(\frac{1}{N+3}\bigr)^n \sum_{k=2}^N c_k h_k^i$, we see from Lemma 6.21, $b_1 = -c_1$ and (6.54) that

$$\begin{aligned}
\frac{\Gamma(h,h)(K_w)}{\Gamma(h,h)(K_v)} &= \frac{\mathcal{E}(h_\tau \circ F_{ij^n}, h_\tau \circ F_{ij^n})}{\mathcal{E}(h_\tau \circ F_{ji^n}, h_\tau \circ F_{ji^n})} = \frac{\bigl(\frac{N+1}{N+3}\bigr)^{2n} b_1^2 + \bigl(\frac{1}{N+3}\bigr)^{2n} \sum_{k=2}^N b_k^2}{\bigl(\frac{N+1}{N+3}\bigr)^{2n} c_1^2 + \bigl(\frac{1}{N+3}\bigr)^{2n} \sum_{k=2}^N c_k^2} \\
&= \frac{((N+1)^{2n} - 1) b_1^2 + \sum_{k=1}^N b_k^2}{((N+1)^{2n} - 1) b_1^2 + \sum_{k=1}^N c_k^2} \\
&\leq \frac{((N+1)^{2n} - 1) b_1^2 + (N+1)^2 \sum_{k=1}^N c_k^2}{((N+1)^{2n} - 1) b_1^2 + \sum_{k=1}^N c_k^2} \leq (N+1)^2,
\end{aligned} \tag{6.55}$$

which proves (6.23) with $\mu = \Gamma(h, h)$ and thereby that $\bigl(K, R_{\mathcal{E}}, \Gamma(h, h)\bigr)$ is VD.

Finally, if $\{h_n\}_{n\in\mathbb{N}} \subset \mathcal{H}_0$ satisfies $\sum_{n\in\mathbb{N}} \mathcal{E}(h_n, h_n) \in (0, \infty)$, then $\sum_{n\in\mathbb{N}} \Gamma(h_n, h_n)$ is a Radon measure on K, (6.22) and (6.23) with $\mu = \sum_{n\in\mathbb{N}} \Gamma(h_n, h_n)$ hold by (6.53) and (6.55), and hence $\bigl(K, R_{\mathcal{E}}, \sum_{n\in\mathbb{N}} \Gamma(h_n, h_n)\bigr)$ is VD. $\square$

Despite Proposition 6.34, Theorem 6.33 does NOT extend to the case of $N \geq 3$, which is the main result of this subsection and stated as follows; recall Proposition 6.26-(3).

Theorem 6.35 *Assume that $N \geq 3$. Then $d_{\mathrm{int}}^{\Gamma(h,h)} \notin \mathcal{J}(K, R_{\mathcal{E}})$ for any $h \in \mathcal{H}_0$. Equivalently, $\mathcal{G}(K, R_{\mathcal{E}}, m, \mathcal{E}, \mathcal{F}) = \emptyset$, i.e., the infimum in (1.4) is not attained for $(K, R_{\mathcal{E}}, m, \mathcal{E}, \mathcal{F})$.*

Theorem 6.35 is proved by deducing a contradiction to the conclusion of the following proposition through taking scaling limits of functions in $\mathcal{H}_0$ on the basis of Proposition 6.17.

Proposition 6.36 *Let $i, j \in S$, $i \neq j$ and let $h^{i,j} \in \mathcal{H}_0$ be such that $h^{i,j}|_{V_0} = \mathbb{1}_{\{q_i\}} - \mathbb{1}_{\{q_j\}}$. Then $d_{\mathrm{int}}^{\Gamma(h^{i,j},h^{i,j})}(x, y) = 0$ for any $k, l \in S \setminus \{i, j\}$ and any $x, y \in \{(1-t)q_k + tq_l \mid t \in [0, 1]\}$. Thus if $N \geq 3$, then $d_{\mathrm{int}}^{\Gamma(h^{i,j},h^{i,j})}$ is not a metric on K and $\Gamma(h^{i,j}, h^{i,j}) \notin \mathcal{G}(K, R_{\mathcal{E}}, m, \mathcal{E}, \mathcal{F})$.*

Proof In the same way as in the proof of Proposition 6.34, it easily follows by $\mathcal{H}_0 \subset \mathcal{H}_1$, (6.11) and solving [60, (3.2.1)] that $h^{i,j} \circ F_k = \frac{1}{N+3} h^{i,j}$ for any $k \in S \setminus \{i, j\}$. Let $u \in \mathcal{F}$ satisfy $\Gamma(u, u) \leq \Gamma(h^{i,j}, h^{i,j})$. Then setting $C := \mathrm{diam}_{R_{\mathcal{E}}}(K)$, we see from Lemma 6.21 that for any $w \in \bigcup_{n=1}^{\infty}(S\setminus\{i, j\})^n$ and any $x, y \in K_w$,

$$
\begin{aligned}
|u(x) - u(y)|^2 &= \left|(u \circ F_w)(F_w^{-1}(x)) - (u \circ F_w)(F_w^{-1}(y))\right|^2 \\
&\leq C\mathcal{E}(u \circ F_w, u \circ F_w) = Cr_w \Gamma(u, u)(K_w) \\
&\leq Cr_w \Gamma(h^{i,j}, h^{i,j})(K_w) = C\mathcal{E}(h^{i,j} \circ F_w, h^{i,j} \circ F_w) \\
&= (N+3)^{-2|w|} C\mathcal{E}(h^{i,j}, h^{i,j}).
\end{aligned}
\tag{6.56}
$$

Now let $k, l \in S \setminus \{i, j\}$ and $x, y \in \{(1-t)q_k + tq_l \mid t \in [0, 1]\}$. Then for any $n \in \mathbb{N}$, taking $v^{(n)}, \tau^{(n)} \in \{k, l\}^n$ such that $x \in K_{v^{(n)}}$ and $y \in K_{\tau^{(n)}}$, from (6.56) we obtain

$$
\begin{aligned}
&|u(x) - u(y)| \\
&\leq \left|u(x) - u(F_{v^{(n)}}(q_k))\right| + \left|u(F_{v^{(n)}}(q_k)) - u(F_{\tau^{(n)}}(q_l))\right| \\
&\quad + \left|u(F_{\tau^{(n)}}(q_l)) - u(y)\right| \\
&\leq 2(N+3)^{-n} C^{1/2} \mathcal{E}(h^{i,j}, h^{i,j})^{1/2} + \sum_{w \in \{k,l\}^n} \left|u(F_w(q_k)) - u(F_w(q_l))\right| \\
&\leq (2 + 2^n)(N+3)^{-n} C^{1/2} \mathcal{E}(h^{i,j}, h^{i,j})^{1/2} \xrightarrow{n \to \infty} 0
\end{aligned}
$$

and thus $u(x) - u(y) = 0$. Since $u \in \mathcal{F}$ satisfying $\Gamma(u, u) \leq \Gamma(h^{i,j}, h^{i,j})$ is arbitrary, it follows from (6.51) with $\mu = \Gamma(h^{i,j}, h^{i,j})$ that $d_{\mathrm{int}}^{\Gamma(h^{i,j},h^{i,j})}(x, y) = 0$. Finally, if $N \geq 3$, then we can choose k, l as above so that $k \neq l$, hence x, y as above so that $x \neq y$, thus $d_{\mathrm{int}}^{\Gamma(h^{i,j},h^{i,j})}$ is not a metric on K by $d_{\mathrm{int}}^{\Gamma(h^{i,j},h^{i,j})}(x, y) = 0$, and therefore $\Gamma(h^{i,j}, h^{i,j}) \notin \mathcal{G}(K, R_{\mathcal{E}}, m, \mathcal{E}, \mathcal{F})$ by Proposition 6.26-(2). □

Proof of Theorem 6.35 Let $h \in \mathcal{H}_0$. By Proposition 6.26-(2),(3) it suffices to show that $\Gamma(h, h) \notin \mathcal{G}(K, R_{\mathcal{E}}, m, \mathcal{E}, \mathcal{F})$. Since this is clear for $h \in \mathbb{R}\mathbb{1}_K$

by $\Gamma(h,h)(K)=\mathcal{E}(h,h)=0$, in the rest of this proof we assume that $h\in\mathcal{H}_0\setminus\mathbb{R}\mathbb{1}_K$. Take $j\in S$ such that $h(q_j)=\min_{q\in V_0}h(q)$, let $\{h^j_k\}_{k=0}^N$ be the basis of $\mathcal{H}_0$ as introduced in the first paragraph of the proof of Proposition 6.34, and set $\psi:=h^j_1$, so that $\psi\circ F_j=\frac{N+1}{N+3}\psi$, for a unique $a\in\mathbb{R}$ we have $(h-a\psi-h(q_j)\mathbb{1}_K)\circ F_j=\frac{1}{N+3}(h-a\psi-h(q_j)\mathbb{1}_K)$, and $a>0$ by $h(q_j)=\min_{q\in V_0}h(q)$. Setting $h_{j^n}:=\mathcal{E}(h\circ F_{j^n},h\circ F_{j^n})^{-1/2}h\circ F_{j^n}$ for $n\in\mathbb{N}\cup\{0\}$, we easily see from these observations and $\mathcal{E}(\psi,\psi)=1$ that $\lim_{n\to\infty}\mathcal{E}(\psi-h_{j^n},\psi-h_{j^n})=0$.

Now suppose that $\Gamma(h,h)\in\mathcal{G}(K,R_{\mathcal{E}},m,\mathcal{E},\mathcal{F})$, which means, by (6.32), (6.35) and (6.33), that $h_{j^0}\in\mathcal{H}_0(\mathcal{G}(\eta,C))$ for some $(\eta,C)\in\mathrm{Homeo}^+\times(1,\infty)$. Then, with $c_{R_{\mathcal{E}}}$ as in Lemma 6.11, $\{h_{j^n}\}_{n\in\mathbb{N}\cup\{0\}}\subset\mathcal{H}_0(\mathcal{G}(c_{R_{\mathcal{E}}}^{-1}\eta,C))$ by Proposition 6.17-(2) and hence $\psi\in\mathcal{H}_0(\mathcal{G}(c_{R_{\mathcal{E}}}^{-1}\eta,C))$ by $\lim_{n\to\infty}\mathcal{E}(\psi-h_{j^n},\psi-h_{j^n})=0$, (6.34) and Proposition 6.17-(1). Further, letting $i\in S\setminus\{j\}$ and setting $\varphi:=\mathcal{E}(\psi\circ F_i,\psi\circ F_i)^{-1/2}\bigl(\psi\circ F_i-N^{-1/2}\frac{N+2}{N+3}\mathbb{1}_K\bigr)$, we would have $\varphi=(2N+2)^{-1/2}h^{i,j}$ with $h^{i,j}$ as in Proposition 6.36 by $\mathcal{H}_0\subset\mathcal{H}_1$, (6.11) and solving [60, (3.2.1)], $\varphi\in\mathcal{H}_0(\mathcal{G}(c_{R_{\mathcal{E}}}^{-2}\eta,C))$ by $\psi\in\mathcal{H}_0(\mathcal{G}(c_{R_{\mathcal{E}}}^{-1}\eta,C))$ and Proposition 6.17-(2), and thus $\Gamma(h^{i,j},h^{i,j})=(2N+2)\Gamma(\varphi,\varphi)\in\mathcal{G}(K,R_{\mathcal{E}},m,\mathcal{E},\mathcal{F})$ by (6.33), (6.32) and (6.35). This would be a contradiction to Proposition 6.36 and thereby proves that $\Gamma(h,h)\notin\mathcal{G}(K,R_{\mathcal{E}},m,\mathcal{E},\mathcal{F})$. □

We conclude this subsection with the following theorem, which is an easy consequence of the conjunction of Proposition 6.34, [64, Corollary 15.12] and Theorem 6.35.

Theorem 6.37 *Assume that $N\geq 3$, let $\{h_n\}_{n\in\mathbb{N}}\subset\mathcal{H}_0$ satisfy $\sum_{n\in\mathbb{N}}\mathcal{E}(h_n,h_n)\in(0,\infty)$ and set $\mu:=\sum_{n\in\mathbb{N}}\Gamma(h_n,h_n)$. Then there does not exist a metric θ on K compatible with the original topology of K such that the MMD space $(K,\theta,\mu,\mathcal{E},\mathcal{F})$ satisfies HKE(2).*

Proof Since $(K,R_{\mathcal{E}},\mu)$ is VD by Proposition 6.34, μ is a Radon measure on K with full support, hence $\mu\in\mathcal{A}(K,R_{\mathcal{E}},m,\mathcal{E},\mathcal{F})$ by (6.12) with m in place of μ, and [64, Corollary 15.12] is applicable to $(K,R_{\mathcal{E}},\mu,\mathcal{E},\mathcal{F})$ and yields $\rho\in\mathcal{J}(K,R_{\mathcal{E}})$ and $\beta\in(1,\infty)$ such that $(K,\rho,\mu,\mathcal{E},\mathcal{F})$ has a continuous heat kernel $p^\mu=p^\mu_t(x,y)\colon(0,\infty)\times K\times K\to\mathbb{R}$ and satisfies VD and HKE(β) with “μ-a.e. x,y” in (4.1) and (4.2) replaced by “any x,y”.

Now suppose that there existed a metric θ on K compatible with the original topology of K such that $(K,\theta,\mu,\mathcal{E},\mathcal{F})$ satisfied HKE(2). Then for each $t\in(0,\infty)$, we easily see from the continuity of p^μ_t, the lower semi-continuity of $\mu(B_\theta(\cdot,t^{1/2}))$, the upper semi-continuity of $\mu\bigl(\overline{B}_\theta(\cdot,t^{1/2})\bigr)$ and VD of (K,ρ,μ) that HKE(2) for $(K,\theta,\mu,\mathcal{E},\mathcal{F})$ would also hold with the same heat kernel p^μ and with “μ-a.e. x,y” in (4.1) and (4.2) replaced by "any x, y". Now for any $x, y \in K$ with $x \neq y$ we would obtain, first $\mu(B_\rho(x, t^{1/\beta}))^{-1} \asymp p_t^\mu(x, x) \asymp \mu(B_\theta(x, t^{1/2}))^{-1}$ for any $t \in (0, \infty)$, then $\rho(x, y)^{\beta/2} \lesssim \theta(x, y)$ by combining (4.1) for ρ, μ, β and (4.2) for $\theta, \mu, 2$ with $t = (\theta(x, y)/\delta)^2$ for a constant $\delta \in (0, \infty)$, and $\theta(x, y) \lesssim \rho(x, y)^{\beta/2}$ by combining (4.1) for $\theta, \mu, 2$ and (4.2) for ρ, μ, β with $t = (\rho(x, y)/\delta')^\beta$ for a constant $\delta' \in (0, \infty)$. Thus $\theta \asymp \rho^{\beta/2}$, in particular $\theta \in \mathcal{J}(K, \rho) = \mathcal{J}(K, R_\mathcal{E})$ by $\rho \in \mathcal{J}(K, R_\mathcal{E})$ and (1.3), and VD of (K, ρ, μ) would imply VD of (K, θ, μ), whence $(K, \theta, \mu, \mathcal{E}, \mathcal{F})$ would satisfy PHI(2) by Theorem 4.5. Therefore we would get $\mu \in \mathcal{G}(K, R_\mathcal{E}, m, \mathcal{E}, \mathcal{F})$, which would contradict Theorem 6.35 and completes the proof. $\square$

6.4 The case of generalized Sierpiński carpets

In this subsection, we treat the case of the canonical self-similar Dirichlet form on an arbitrary generalized Sierpiński carpet and see that the arguments in Sect. 6.2 go through also in this case with just slight modifications in the proofs. We closely follow the presentation of [54, Section 4] for the introduction of the framework of this subsection up to Theorem 6.48 below, which we repeat here for the reader's convenience.

We fix the following setting throughout this subsection.

Framework 6.38 Let $N, l \in \mathbb{N}$, $N \geq 2$, $l \geq 3$ and set $Q_0 := [0, 1]^N$. Let $S \subsetneq \{0, 1, \ldots, l-1\}^N$ be non-empty, define $f_i \colon \mathbb{R}^N \to \mathbb{R}^N$ by $f_i(x) := l^{-1}i + l^{-1}x$ for each $i \in S$ and set $Q_1 := \bigcup_{i \in S} f_i(Q_0)$, so that $Q_1 \subsetneq Q_0$. Let K be the self-similar set associated with $\{f_i\}_{i \in S}$, i.e., the unique non-empty compact subset of $\mathbb{R}^N$ such that $K = \bigcup_{i \in S} f_i(K)$, which exists and satisfies $K \subsetneq Q_0$ thanks to $Q_1 \subsetneq Q_0$ by [60, Theorem 1.1.4], and set $F_i := f_i|_K$ for each $i \in S$, so that $\mathrm{GSC}(N, l, S) := (K, S, \{F_i\}_{i \in S})$ is a self-similar structure by [60, Theorem 1.2.3]. Let $d \colon K \times K \to [0, \infty)$ be the Euclidean metric on K given by $d(x, y) := |x - y|$, set $d_\mathrm{f} := \log_l \#S$, and let m be the self-similar measure on $\mathrm{GSC}(N, l, S)$ with weight $(1/\#S)_{i \in S}$, i.e., the unique Borel probability measure on K such that $m(K_w) = (\#S)^{-|w|}$ for any $w \in W_*$, which exists by [60, Propositions 1.5.8, 1.4.3, 1.4.4 and Corollary 1.4.8].

Recall that d_f is the Hausdorff dimension of (K, d) and that m is a constant multiple of the d_f-dimensional Hausdorff measure on (K, d); see, e.g., [60, Proposition 1.5.8 and Theorem 1.5.7]. Note that $d_\mathrm{f} < N$ by $S \subsetneq \{0, 1, \ldots, l-1\}^N$.

The following definition is essentially due to Barlow and Bass [5, Section 2]. In what follows, the interior of $A \subset \mathbb{R}^N$ in $\mathbb{R}^N$ is denoted by $\mathrm{int}_{\mathbb{R}^N}(A)$.

Definition 6.39 (Generalized Sierpiński carpet, [8, Subsection 2.2]) $\mathrm{GSC}(N, l, S)$ is called a *generalized Sierpiński carpet* if and only if the following four conditions are satisfied:

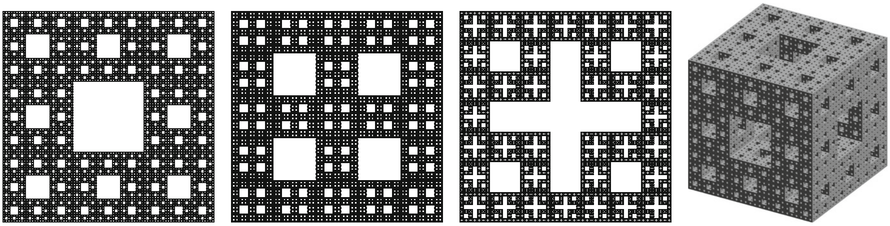

Fig. 2 Sierpiński carpet, some other generalized Sierpiński carpets with $N = 2$ and Menger sponge

(GSC1) (Symmetry) $f(Q_1) = Q_1$ for any isometry f of $\mathbb{R}^N$ with $f(Q_0) = Q_0$.
(GSC2) (Connectedness) Q_1 is connected.
(GSC3) (Non-diagonality) $\mathrm{int}_{\mathbb{R}^N}\bigl(Q_1 \cap \prod_{k=1}^{N}[(i_k - \varepsilon_k)l^{-1}, (i_k + 1)l^{-1}]\bigr)$ is either empty or connected for any $(i_k)_{k=1}^N \in \mathbb{Z}^N$ and any $(\varepsilon_k)_{k=1}^N \in \{0, 1\}^N$.
(GSC4) (Borders included) $[0, 1] \times \{0\}^{N-1} \subset Q_1$.

As special cases of Definition 6.39, $\mathrm{GSC}(2, 3, S_{\mathrm{SC}})$ and $\mathrm{GSC}(3, 3, S_{\mathrm{MS}})$ are called the *(two-dimensional standard) Sierpiński carpet* and the *Menger sponge*, respectively, where $S_{\mathrm{SC}} := \{0, 1, 2\}^2 \setminus \{(1, 1)\}$ and $S_{\mathrm{MS}} := \bigl\{(i_1, i_2, i_3) \in \{0, 1, 2\}^3 \bigm| \sum_{k=1}^{3} \mathbb{1}_{\{1\}}(i_k) \leq 1\bigr\}$ (see Fig. 2 above).

See [5, Remark 2.2] for a description of the meaning of each of the four conditions (GSC1), (GSC2), (GSC3) and (GSC4) in Definition 6.39. We remark that there are several equivalent ways of stating the non-diagonality condition, as in the following proposition.

Proposition 6.40 ([51, §2]) *Set* $|x|_1 := \sum_{k=1}^{N}|x_k|$ *for* $x = (x_k)_{k=1}^N \in \mathbb{R}^N$. *Then* (GSC3) *is equivalent to any one of the following three conditions:*

$(\mathrm{ND})_{\mathbb{N}}$ $\mathrm{int}_{\mathbb{R}^N}\bigl(Q_1 \cap \prod_{k=1}^{N}[(i_k - 1)l^{-n}, (i_k + 1)l^{-n}]\bigr)$ *is either empty or connected for any* $n \in \mathbb{N}$ *and any* $(i_k)_{k=1}^N \in \{1, 2, \dots, l^n - 1\}^N$.
$(\mathrm{ND})_2$ $\mathrm{int}_{\mathbb{R}^N}\bigl(Q_1 \cap \prod_{k=1}^{N}[(i_k - 1)l^{-2}, (i_k + 1)l^{-2}]\bigr)$ *is either empty or connected for any* $(i_k)_{k=1}^N \in \{1, 2, \dots, l^2 - 1\}^N$.
(NDF) *For any* $i, j \in S$ *with* $f_i(Q_0) \cap f_j(Q_0) \neq \emptyset$ *there exists* $\{n(k)\}_{k=0}^{|i-j|_1} \subset S$ *such that* $n(0) = i$, $n(|i - j|_1) = j$ *and* $|n(k) - n(k + 1)|_1 = 1$ *for any* $k \in \{0, \dots, |i - j|_1 - 1\}$.

Remark 6.41 (1) Only the case of $n = 1$ of $(\mathrm{ND})_{\mathbb{N}}$ had been assumed in the original definition of generalized Sierpiński carpets in [5, Section 2], but Barlow, Bass, Kumagai and Teplyaev [8] later realized that it had been too weak for [5, Proof of Theorem 3.19] and had to be replaced by $(\mathrm{ND})_{\mathbb{N}}$ (or equivalently, by (GSC3)).

(2) In fact, [8, Subsection 2.2] assumes instead of (GSC2) the seemingly stronger condition that $\mathrm{int}_{\mathbb{R}^N} Q_1$ is connected, but it is implied by (GSC2) and (GSC3) in view of (NDF) in Proposition 6.40 and is thus equivalent to (GSC2) under the assumption of (GSC3).

In the rest of this subsection, we assume that $\mathcal{L} := \mathrm{GSC}(N, l, S) = (K, S, \{F_i\}_{i \in S})$ is a generalized Sierpiński carpet. Then we easily see the following proposition and lemma.

Proposition 6.42 *Set* $S_{k,\varepsilon} := \{(i_n)_{n=1}^N \in S \mid i_k = (l-1)\varepsilon\}$ *for* $k \in \{1, 2, \dots, N\}$ *and* $\varepsilon \in \{0, 1\}$. *Then* $\mathcal{P}_{\mathcal{L}} = \bigcup_{k=1}^N (S_{k,0}^{\mathbb{N}} \cup S_{k,1}^{\mathbb{N}})$ *and* $\overline{V_0} = V_0 = K \setminus (0,1)^N \neq K$.

Lemma 6.43 *There exist* $c_1, c_2 \in (0, \infty)$ *such that for any* $(x, s) \in K \times (0, \mathrm{diam}_d(K)]$,

$$c_1 s^{d_{\mathrm{f}}} \leq m(B_d(x, s)) \leq c_2 s^{d_{\mathrm{f}}}. \tag{6.57}$$

We next recall some basics of the canonical self-similar Dirichlet form on $\mathrm{GSC}(N, l, S)$. There are two established ways of constructing a non-degenerate m-symmetric diffusion without killing inside on K, or equivalently, a non-zero, strongly local, regular symmetric Dirichlet form on $L^2(K, m)$, one by Barlow and Bass [4,5] using the reflecting Brownian motions on the domains approximating K, and the other by Kusuoka and Zhou [70] based on graph approximations. It had been a long-standing open problem to prove that the constructions in [4,5] and in [70] give rise to the same diffusion on K, which Barlow, Bass, Kumagai and Teplyaev [8] have finally solved by proving the uniqueness of a non-zero conservative regular symmetric Dirichlet form on $L^2(K, m)$ possessing certain local symmetry. As a consequence of the results in [8], after some additional arguments in [47,53,55] we have the unique existence of a canonical self-similar Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(K, m)$, the heat kernel estimates HKE(d_{w}) with $d_{\mathrm{w}} > 2$, and hence PHI(d_{w}) by Lemma 6.43 and Theorem 4.5, as follows.

Definition 6.44 We define

$$\mathcal{I}_0 := \{f|_K \mid f \text{ is an isometry of } \mathbb{R}^N,\ f(Q_0) = Q_0\}, \tag{6.58}$$

which forms a subgroup of the group of isometries of (K, d) by virtue of (GSC1).

Theorem 6.45 ([8, Theorems 1.2 and 4.32], [47, Proposition 5.1], [53, Proposition 5.9]) *There exists a unique (up to constant multiples of* $\mathcal{E}$*) regular symmetric Dirichlet form* $(\mathcal{E}, \mathcal{F})$ *on* $L^2(K, m)$ *satisfying* $\mathcal{E}(u, u) > 0$ *for* *some* $u \in \mathcal{F}$, $\mathbb{1}_K \in \mathcal{F}$, $\mathcal{E}(\mathbb{1}_K, \mathbb{1}_K) = 0$, (6.7), (6.8) *for some* $r \in (0, \infty)$, *and the following:*

(GSCDF) *If* $u \in \mathcal{F} \cap C(K)$ *and* $g \in \mathcal{I}_0$ *then* $u \circ g \in \mathcal{F}$ *and* $\mathcal{E}(u \circ g, u \circ g) = \mathcal{E}(u, u)$.

Throughout the rest of this subsection, we fix $(\mathcal{E}, \mathcal{F})$ and r as given in Theorem 6.45; note that r is uniquely determined by $(\mathcal{E}, \mathcal{F})$, since $\mathcal{E}(u, u) > 0$ for some $u \in \mathcal{F} \cap C(K)$ by the existence of such $u \in \mathcal{F}$ and the denseness of $\mathcal{F} \cap C(K)$ in the Hilbert space $(\mathcal{F}, \mathcal{E}_1 := \mathcal{E} + \langle \cdot, \cdot \rangle_{L^2(K,m)})$.

Definition 6.46 The regular symmetric Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(K, m)$ as in Theorem 6.45 is called the *canonical Dirichlet form* on $\mathrm{GSC}(N, l, S)$, and we set $d_{\mathrm{w}} := \log_l(\#S/r)$. Note that $(\mathcal{E}, \mathcal{F})$ is also strongly local by the same argument as [45, Proof of Lemma 3.12] based on (6.7), (6.8) and $\mathcal{E}(\mathbb{1}_K, \mathbb{1}_K) = 0$.

Theorem 6.47 ([5, Remarks 5.4-1.], [55, Theorem 2.7]) $d_{\mathrm{w}} > 2$.

Theorem 6.48 ([5, Theorem 1.3], [8, Theorem 4.30 and Remark 4.33]) *There exists a (unique) continuous version* $p = p_t(x, y)\colon (0, \infty) \times K \times K \to [0, \infty)$ *of the heat kernel of* $(K, d, m, \mathcal{E}, \mathcal{F})$, *and there exist* $c_1, c_2, c_3, c_4 \in (0, \infty)$ *such that for any* $(t, x, y) \in (0, 1] \times K \times K$,

$$\frac{c_1}{t^{d_{\mathrm{f}}/d_{\mathrm{w}}}} \exp\biggl(-\Bigl(\frac{d(x, y)^{d_{\mathrm{w}}}}{c_2 t}\Bigr)^{\frac{1}{d_{\mathrm{w}}-1}}\biggr) \leq p_t(x, y) \leq \frac{c_3}{t^{d_{\mathrm{f}}/d_{\mathrm{w}}}} \exp\biggl(-\Bigl(\frac{d(x, y)^{d_{\mathrm{w}}}}{c_4 t}\Bigr)^{\frac{1}{d_{\mathrm{w}}-1}}\biggr). \tag{6.59}$$

In particular, $(K, d, m, \mathcal{E}, \mathcal{F})$ *satisfies* $\mathrm{HKE}(d_{\mathrm{w}})$ *and* $\mathrm{PHI}(d_{\mathrm{w}})$ *by Lemma* 6.43 *and Theorem* 4.5.

The following lemma is an immediate consequence of Theorem 6.48. Recall that $\mathcal{F}$ is a Hilbert space under the inner product $\mathcal{E}_1 := \mathcal{E} + \langle \cdot, \cdot \rangle_{L^2(K,m)}$.

Lemma 6.49 *The inclusion map* $\mathcal{F} \hookrightarrow L^2(K, m)$ *is a compact linear operator from* $(\mathcal{F}, \mathcal{E}_1)$ *to* $L^2(K, m)$, *and there exists* $C_{\mathrm{P}} \in (0, \infty)$ *such that for any* $u \in \mathcal{F}$,

$$\int_K \Bigl| u - \int_K u \, dm \Bigr|^2 dm \leq C_{\mathrm{P}} \mathcal{E}(u, u). \tag{6.60}$$

In particular, $\{u \in \mathcal{F} \mid \mathcal{E}(u, u) = 0\} = \mathbb{R}\mathbb{1}_K$, $(\mathcal{F}/\mathbb{R}\mathbb{1}_K, \mathcal{E})$ *is a Hilbert space, and the extended Dirichlet space* $\mathcal{F}_e$ *of* $(K, d, m, \mathcal{E}, \mathcal{F})$ *coincides with* $\mathcal{F}$.

Proof The compactness of the inclusion map $\mathcal{F} \hookrightarrow L^2(K, m)$ follows from Theorem 6.48 and [30, Corollary 4.2.3 and Exercise 4.2]. The existence of $C_{\mathrm{P}} \in (0, \infty)$ satisfying (6.60) for any $u \in \mathcal{F}$ is implied by Theorems 6.48 and 4.5 as a special case of $\mathrm{PI}(d_{\mathrm{w}})$, or more elementarily by [30, Theorems 4.5.1 and 4.5.3] and the fact that $\{u \in \mathcal{F} \mid \mathcal{E}(u, u) = 0\} = \mathbb{R}\mathbb{1}_K$ by $\mathcal{E}(\mathbb{1}_K, \mathbb{1}_K) = 0$, Theorem 6.48 and [24, Theorem 2.1.11]. The completeness of $(\mathcal{F}/\mathbb{R}\mathbb{1}_K, \mathcal{E})$ is immediate from that of $(\mathcal{F}, \mathcal{E}_1)$ and (6.60), and they also easily imply the equality $\mathcal{F}_e = \mathcal{F}$ as proved in [48, Proposition 2.9]. □

Introducing the following space of $\mathcal{E}$-harmonic functions is convenient for our purpose in this subsection; recall (2.4) for the definition of $\mathcal{E}$-harmonicity of functions.

Definition 6.50 We define

$$\mathcal{H}_0 := \left\{ h \in \mathcal{F} \;\middle|\; \begin{array}{l} h \text{ is } \mathcal{E}\text{-harmonic on } K \setminus V_0, \text{ i.e., } \mathcal{E}(h, v) = 0 \\ \text{for any } v \in \mathcal{F} \cap C(K) \text{ with } \operatorname{supp}_m[v] \subset K \setminus V_0 \end{array} \right\}, \tag{6.61}$$

which is clearly a linear subspace of $\mathcal{F}$ and satisfies $\mathbb{R}\mathbb{1}_K \subset \mathcal{H}_0$ by $\mathcal{E}(\mathbb{1}_K, \mathbb{1}_K) = 0$. Note that $\mathcal{H}_0$ is weakly closed in $(\mathcal{F}, \mathcal{E}_1)$ since $\mathcal{E}(\cdot, v)$ is a bounded linear functional on $(\mathcal{F}, \mathcal{E}_1)$ for any $v \in \mathcal{F}$.

As the counterpart of Proposition 6.10 for generalized Sierpiński carpets, we have the following characterization of the pair (θ, μ) of $\theta \in \mathcal{J}(K, d)$ (recall (1.2)) and $\mu \in \mathcal{A}(K, d, m, \mathcal{E}, \mathcal{F})$ (recall Definition 2.8) such that $(K, \theta, \mu, \mathcal{E}^\mu, \mathcal{F}^\mu)$ satisfies $\mathrm{PHI}(\beta)$; see also [66] for related results.

Theorem 6.51 *Let $\theta \in \mathcal{J}(K, d)$, let μ be a Radon measure on K with full support and let $\beta \in (1, \infty)$. Then the following conditions are equivalent:*

(a) $\mu \in \mathcal{A}(K, d, m, \mathcal{E}, \mathcal{F})$ and $(K, \theta, \mu, \mathcal{E}^\mu, \mathcal{F}^\mu)$ satisfies $\mathrm{PHI}(\beta)$.
(b) There exists $C \in (1, \infty)$ such that for any $w \in W_*$, with $r_w := r^{|w|}$,

$$C^{-1}(\operatorname{diam}_\theta(K_w))^\beta \leq r_w \mu(K_w) \leq C(\operatorname{diam}_\theta(K_w))^\beta. \tag{6.62}$$

Moreover, if either of these conditions holds, then $\mu(F_w(V_0)) = 0$ for any $w \in W_*$ and $\mu(\{x\}) = 0$ for any $x \in K$.

Theorem 6.51 follows by repeating the same arguments as the proof of Proposition 6.10, on the basis of the following proposition concluded from Theorem 4.5 with the help of [9, Lemma 5.22 and Proposition 6.16].

Proposition 6.52 *Let $\theta \in \mathcal{J}(K, d)$, let μ be a Radon measure on K with full support and let $\beta \in (1, \infty)$. Then Theorem 6.51-(a) is equivalent to the following condition:*

(c) (K, θ, μ) *is* VD *and there exists* $C \in (0, \infty)$ *such that for any* $x, y \in K$ *with* $x \neq y$,

$$C^{-1}\theta(x, y)^{\beta} \leq d(x, y)^{d_{\mathrm{w}}-d_{\mathrm{f}}}\mu\bigl(B_{\theta}(x, \theta(x, y))\bigr) \leq C\theta(x, y)^{\beta}. \quad (6.63)$$

Proof Note first that, if μ is admissible with respect to $(K, d, m, \mathcal{E}, \mathcal{F})$, then for any Borel subsets A, B of K with B closed in K and $\overline{A} \cap B = \emptyset$, by [24, Theorem 5.2.11] and [33, Theorem 4.6.2 and Lemma 2.1.4] we have

$$\begin{aligned}
&\mathrm{Cap}^{\mu}(A, B)\\
&\quad = \inf\{\mathcal{E}(f^{+} \wedge 1, f^{+} \wedge 1) \mid f \in \mathcal{F}^{\mu}, \overline{A} \cap \mathrm{supp}_{\mu}[f - \mathbb{1}_{K}] = \emptyset = B \cap \mathrm{supp}_{\mu}[f]\}\\
&\quad = \inf\left\{\mathcal{E}(f, f) \,\middle|\, \begin{array}{l} f \in \mathcal{F}^{\mu}, 0 \leq f \leq 1\ \mu\text{-a.e.}, \\ \overline{A} \cap \mathrm{supp}_{\mu}[f - \mathbb{1}_{K}] = \emptyset = B \cap \mathrm{supp}_{\mu}[f] \end{array}\right\}\\
&\quad = \inf\{\mathcal{E}(f, f) \mid f \in \mathcal{F}(A, B), 0 \leq f \leq 1\ m\text{-a.e.}\}\\
&\quad = \inf\{\mathcal{E}(f^{+} \wedge 1, f^{+} \wedge 1) \mid f \in \mathcal{F}(A, B)\} = \mathrm{Cap}(A, B), \qquad (6.64)
\end{aligned}$$

where $\mathrm{Cap}^{\mu}(A, B)$ and $\mathrm{Cap}(A, B)$ denote the capacity between A, B with respect to $(K, \theta, \mu, \mathcal{E}^{\mu}, \mathcal{F}^{\mu})$ and $(K, d, m, \mathcal{E}, \mathcal{F})$, respectively. Next, since $(K, d, m, \mathcal{E}, \mathcal{F})$ satisfies (6.57), cap(d_{w}) and EHI by Lemma 6.43, Theorems 6.48 and 4.5, there exist $C_1, A_1, A_2 \in (1, \infty)$ with $A_2 \geq 2$ such that for any $(x, s) \in K \times (0, \mathrm{diam}_d(K)/A_2)$,

$$C_1^{-1}s^{d_{\mathrm{f}}-d_{\mathrm{w}}} \leq \mathrm{Cap}\bigl(B_d(x, s), K \setminus B_d(x, A_1 s)\bigr) \leq C_1 s^{d_{\mathrm{f}}-d_{\mathrm{w}}}, \qquad (6.65)$$

and by the quasisymmetry of θ to d and Lemma 4.8 we have EHI for $(K, \theta, m, \mathcal{E}, \mathcal{F})$, as well as for $(K, \theta, \mu, \mathcal{E}^{\mu}, \mathcal{F}^{\mu})$ provided $\mu \in \mathcal{A}(K, d, m, \mathcal{E}, \mathcal{F})$.

To prove the desired equivalence, assume Theorem 6.51-(a), so that $(K, \theta, \mu, \mathcal{E}^{\mu}, \mathcal{F}^{\mu})$ satisfies VD and cap(β) by Theorem 4.5 and therefore in view of (6.64) there exist $C_2, A_3, A_4 \in (1, \infty)$ such that

$$C_2^{-1}\frac{\mu(B_{\theta}(x, s))}{s^{\beta}} \leq \mathrm{Cap}\bigl(B_{\theta}(x, s), K \setminus B_{\theta}(x, A_3 s)\bigr) \leq C_2\frac{\mu(B_{\theta}(x, s))}{s^{\beta}} \qquad (6.66)$$

for any $(x, s) \in K \times (0, \mathrm{diam}_{\theta}(K)/A_4)$. To verify (6.63), let $x, y \in K$ satisfy $x \neq y$. By the quasisymmetry of θ to d, (3.3) and (3.5), there exist $A_5 \in (1, \infty)$ determined solely by d, θ, A_1 and $A_6 \in (1, \infty)$ determined solely by d, θ, A_3 such that

$$B_{\theta}(x, \theta(x, y)/A_5) \subset B_d(x, d(x, y)) \subset B_d(x, A_1 d(x, y)) \subset B_{\theta}(x, A_5\theta(x, y)), \qquad (6.67)$$

$$B_d(x, d(x,y)/A_6) \subset B_\theta(x, \theta(x,y)) \subset B_\theta(x, A_3\theta(x,y)) \subset B_d(x, A_6 d(x,y)). \tag{6.68}$$

Then by EHI for $(K, d, m, \mathcal{E}, \mathcal{F})$ and $(K, \theta, m, \mathcal{E}, \mathcal{F})$, Remark 2.6 and [9, Lemma 5.22] (note also [9, Theorem 5.4 and Lemma 5.2-(e)]), there exist $A_7 \in (A_2, \infty)$ and $A_8 \in (A_4, \infty)$ such that for any $(z, s) \in K \times (0, \infty)$,

$$\mathrm{Cap}\big(B_d(z,s), K \setminus B_d(z, A_1 s)\big) \asymp \mathrm{Cap}\big(B_d(z,s), K \setminus B_d(z, A_6^2 s)\big) \quad \text{if } s < \mathrm{diam}_d(K)/A_7, \tag{6.69}$$

$$\mathrm{Cap}\big(B_\theta(z,s), K \setminus B_\theta(z, A_3 s)\big) \asymp \mathrm{Cap}\big(B_\theta(z,s), K \setminus B_\theta(z, A_5^2 s)\big) \quad \text{if } s < \mathrm{diam}_\theta(K)/A_8. \tag{6.70}$$

(To be precise, the definition of capacity between sets in [9, Section 5] is slightly different from ours, but they are easily seen to be equivalent to each other by virtue of [33, Lemma 2.2.7-(ii)].) Moreover, the quasisymmetry of θ to d again and (3.7) show that by taking A_8 large enough we may further assume that $\theta(x,y) < \mathrm{diam}_\theta(K)/A_8$ implies $d(x,y) < \mathrm{diam}_d(K)/A_7$. Now, if $\theta(x,y) \geq \mathrm{diam}_\theta(K)/A_8$, then (6.63) clearly holds for some sufficiently large $C \in (0,\infty)$ independent of x, y since $\mu\big(B_\theta(x, \theta(x,y))\big) \asymp \mu(K)$ by VD of (K, θ, μ) and $d(x,y) \asymp \mathrm{diam}_d(K)$ by the quasisymmetry of θ to d and (3.7). Otherwise $\theta(x,y) < \mathrm{diam}_\theta(K)/A_8$, which implies $d(x,y) < \mathrm{diam}_d(K)/A_7$, hence

$$\begin{aligned} C_2 \frac{\mu\big(B_\theta(x, \theta(x,y))\big)}{\theta(x,y)^\beta} &\geq \mathrm{Cap}\big(B_\theta(x, \theta(x,y)), K \setminus B_\theta(x, A_3\theta(x,y))\big) \\ &\geq \mathrm{Cap}\big(B_d(x, d(x,y)/A_6), K \setminus B_d(x, A_6 d(x,y))\big) \\ &\asymp d(x,y)^{d_\mathrm{f} - d_\mathrm{w}} \end{aligned}$$

by (6.66), (6.68), (6.69) and (6.65), and similarly

$$\begin{aligned} C_1 d(x,y)^{d_\mathrm{f} - d_\mathrm{w}} &\geq \mathrm{Cap}\big(B_d(x, d(x,y)), K \setminus B_d(x, A_1 d(x,y))\big) \\ &\geq \mathrm{Cap}\big(B_\theta(x, \theta(x,y)/A_5), K \setminus B_\theta(x, A_5\theta(x,y))\big) \\ &\asymp \frac{\mu\big(B_\theta(x, \theta(x,y))\big)}{\theta(x,y)^\beta} \end{aligned}$$

by (6.65), (6.67), (6.70), (6.66) and VD of (K, θ, μ), proving (6.63) and thereby (c).

Conversely, assume (c). Since K is connected, VD of (K, θ, μ) implies RVD of (K, θ, μ) by Remark 3.18-(b). Note that (6.67) remains valid and that for each $A_3, A_4 \in (1, \infty)$ we still have (6.68), (6.69) and (6.70). Note also that by the connectedness of K and [40, Theorem 11.3] there exist

$\lambda, \alpha \in [1, \infty)$ such that θ is $\eta_{\alpha,\lambda}$-quasisymmetric to d with $\eta_{\alpha,\lambda}$ as in Definition 3.1. Let $x \in K$ and let $s_1, s_2 \in (0, \mathrm{diam}_d(K)/A_2)$ satisfy $s_1 \leq s_2$. Then $d(x, y) = s_1$ and $d(x, z) = s_2$ for some $y, z \in K$ by the connectedness of K, $\mu(B_d(x, s_1)) \asymp \mu\bigl(B_\theta(x, \theta(x, y))\bigr)$ and $\mu(B_d(x, s_2)) \asymp \mu\bigl(B_\theta(x, \theta(x, z))\bigr)$ by (6.67) and VD of (K, θ, μ), and therefore by (6.63), (6.65), the $\eta_{\alpha,\lambda}$-quasisymmetry of θ to d and (3.7),

$$\begin{aligned}
&\frac{\mu(B_d(x, s_2))\,\mathrm{Cap}\bigl(B_d(x, s_1), K \setminus B_d(x, A_1 s_1)\bigr)}{\mu(B_d(x, s_1))\,\mathrm{Cap}\bigl(B_d(x, s_2), K \setminus B_d(x, A_1 s_2)\bigr)} \\
&\quad\asymp \frac{d(x, z)^{d_{\mathrm{w}}-d_{\mathrm{f}}}\mu\bigl(B_\theta(x, \theta(x, z))\bigr)}{d(x, y)^{d_{\mathrm{w}}-d_{\mathrm{f}}}\mu\bigl(B_\theta(x, \theta(x, y))\bigr)} \\
&\quad\asymp \left(\frac{\theta(x, z)}{\theta(x, y)}\right)^{\beta} \in \left[\frac{1}{2\lambda}\left(\frac{s_2}{s_1}\right)^{1/\alpha}, 2^\alpha\lambda\left(\frac{s_2}{s_1}\right)^{\alpha}\right].
\end{aligned} \tag{6.71}$$

It follows from EHI for $(K, d, m, \mathcal{E}, \mathcal{F})$, Remark 2.6 and (6.71) that [9, Proposition 6.16] is applicable to μ and implies that $\mu \in \mathcal{A}(K, d, m, \mathcal{E}, \mathcal{F})$. In particular, $(K, \theta, \mu, \mathcal{E}^\mu, \mathcal{F}^\mu)$ satisfies EHI by Lemma 4.8. Finally, to show cap(β) for $(K, \theta, \mu, \mathcal{E}^\mu, \mathcal{F}^\mu)$, let $A_3, A_4 \in (1, \infty)$ satisfy $A_4 \geq 2$, and choose $A_6, A_7, A_8 \in (1, \infty)$ with $A_7 > A_2$ and $A_8 > A_4$ so that (6.68), (6.69) and (6.70) hold. Again thanks to the quasisymmetry of θ to d and (3.7), by taking A_8 large enough we may further assume that $d(x, y) < \mathrm{diam}_d(K)/A_7$ for any $x, y \in K$ with $\theta(x, y) < \mathrm{diam}_\theta(K)/A_8$. Let $(x, s) \in K \times (0, \mathrm{diam}_\theta(K)/(A_5 A_8))$, so that $\theta(x, y) = s = \theta(x, z)/A_5$ for some $y, z \in K$ by the connectedness of K. Then by (6.68), (6.69), (6.65) and (6.63),

$$\begin{aligned}
&\mathrm{Cap}\bigl(B_\theta(x, s), K \setminus B_\theta(x, A_3 s)\bigr) \\
&\quad\geq \mathrm{Cap}\bigl(B_d(x, d(x, y)/A_6), K \setminus B_d(x, A_6 d(x, y))\bigr) \\
&\quad\asymp d(x, y)^{d_{\mathrm{f}}-d_{\mathrm{w}}} \asymp \frac{\mu\bigl(B_\theta(x, \theta(x, y))\bigr)}{\theta(x, y)^\beta} = \frac{\mu\bigl(B_\theta(x, s)\bigr)}{s^\beta},
\end{aligned}$$

and similarly by (6.70), (6.67), (6.65), (6.63) and VD of (K, θ, μ),

$$\begin{aligned}
&\mathrm{Cap}\bigl(B_\theta(x, s), K \setminus B_\theta(x, A_3 s)\bigr) \\
&\quad\asymp \mathrm{Cap}\bigl(B_\theta(x, \theta(x, z)/A_5), K \setminus B_\theta(x, A_5\theta(x, z))\bigr) \\
&\quad\leq \mathrm{Cap}\bigl(B_d(x, d(x, z)), K \setminus B_d(x, A_1 d(x, z))\bigr) \\
&\quad\asymp d(x, z)^{d_{\mathrm{f}}-d_{\mathrm{w}}} \asymp \frac{\mu\bigl(B_\theta(x, \theta(x, z))\bigr)}{\theta(x, z)^\beta} \asymp \frac{\mu\bigl(B_\theta(x, s)\bigr)}{s^\beta},
\end{aligned}$$

proving (6.66) for $(x,s) \in K \times (0, \mathrm{diam}_\theta(K)/(A_5 A_8))$, namely cap($\beta$) for $(K, \theta, \mu, \mathcal{E}^\mu, \mathcal{F}^\mu)$. Now $(K, \theta, \mu, \mathcal{E}^\mu, \mathcal{F}^\mu)$ satisfies PHI(β) by Theorem 4.5, showing Theorem 6.51-(a). □

Proof of Theorem 6.51 By Proposition 6.52, it suffices to prove the equivalence of Proposition 6.52-(c) and (b). We can verify it in exactly the same way as the proof of Proposition 6.10, by considering the scale $\mathscr{S} = \{\Lambda_s\}_{s\in(0,1]}$ on Σ defined by $\Lambda_1 := \{\emptyset\}$ and

$$\Lambda_s := \{w \mid w = w_1 \dots w_n \in W_* \setminus \{\emptyset\}, l^{1-|w|} > s \geq l^{-|w|}\} \tag{6.72}$$

for each $s \in (0,1)$, which clearly satisfies (6.21), and by using instead of $R_\mathcal{E}$ the Euclidean metric d on K, which is easily seen to satisfy (6.18) with d in place of $R_\mathcal{E}$; note that since (6.19) needs to be replaced by (6.63) we also need to replace $R_\mathcal{E}(x,y)\mu\bigl(B_{R_\mathcal{E}}(x, R_\mathcal{E}(x,y))\bigr)$ in (6.25) by $d(x,y)^{d_\mathrm{w}-d_\mathrm{f}}\mu\bigl(B_d(x,d(x,y))\bigr)$. □

By virtue of Theorem 6.51, the whole of Sect. 6.2 can be easily adapted for the present case, and below we explicitly give the details of the adaptation for concreteness. We begin with stating the main result of this subsection, which requires the following definition. Recall (6.29) for Homeo^+ and Definition 6.50 for $\mathcal{H}_0$.

Definition 6.53 We define $\mathcal{P}(K)$ by (6.30), equip $\mathcal{P}(K)$ with the topology of weak convergence, and for each $(\eta, C) \in \mathrm{Homeo}^+ \times (1,\infty)$ we define $\mathcal{G}(\eta, C) = \mathcal{G}_{N,l,S}(\eta, C)$ by

$$\mathcal{G}(\eta, C) := \left\{ (\theta, \mu) \;\middle|\; \begin{array}{l} \theta \text{ is a metric on } K \text{ and } \eta\text{-quasisymmetric to } d,\ \mu \in \\ \mathcal{P}(K),\ C^{-1} \leq r_w \mu(K_w)/(\mathrm{diam}_\theta(K_w))^2 \leq C \text{ for any} \\ w \in W_* \end{array} \right\}, \tag{6.73}$$

which is considered as a subset of $C(K \times K) \times \mathcal{P}(K)$. We also set

$$\mathcal{G} := \mathcal{G}_{N,l,S} := \bigcup_{(\eta,C)\in \mathrm{Homeo}^+ \times(1,\infty)} \mathcal{G}(\eta, C) \tag{6.74}$$

and for each subset $\mathcal{Z}$ of $\mathcal{G}$ define $\mathcal{H}_0(\mathcal{Z}) \subset \mathcal{H}_0$ by (6.33) and $\widetilde{\mathcal{H}}_0(\mathcal{Z}) \subset \mathcal{H}_0/\mathbb{R}\mathbb{1}_K$ by (6.34).

Since μ is a Radon measure on K with full support for any $(\theta, \mu) \in \mathcal{G}$, it follows from Theorem 6.51 with $\beta = 2$ and (1.3) that

$$\mathcal{G}(K, d, m, \mathcal{E}, \mathcal{F}) = \{a\mu \mid (\theta, \mu) \in \mathcal{G}, a \in (0,\infty)\}. \tag{6.75}$$

In particular,

$$\mathcal{G}(K,d,m,\mathcal{E},\mathcal{F}) \neq \emptyset, \text{ i.e., the infimum in (1.4) is attained for } (K,d,m,\mathcal{E},\mathcal{F}), \text{ if and only if } \mathcal{G} \neq \emptyset, \text{ namely } \mathcal{G}(\eta,C) \neq \emptyset \text{ for some } (\eta,C) \in \mathrm{Homeo}^{+} \times (1,\infty). \tag{6.76}$$

It turns out that in this case $\mathcal{H}_0(\mathcal{G}) \neq \emptyset$, i.e., $(\theta_h, \Gamma(h,h)) \in \mathcal{G}$ for some $h \in \mathcal{H}_0$ and some $\theta_h \in \mathcal{J}(K,d)$, which is the main result of this subsection and stated as follows. *We take arbitrary* $(\eta, C) \in \mathrm{Homeo}^{+} \times (1,\infty)$, *define* $\tilde{\eta} \in \mathrm{Homeo}^{+}$ *by* $\tilde{\eta}(t) := 1/\eta^{-1}(t^{-1})$ $(\tilde{\eta}(0) := 0)$ *and fix them throughout the rest of this subsection.*

Theorem 6.54 *If* $\mathcal{G}(\eta,C) \neq \emptyset$, *then* $\mathcal{H}_0(\mathcal{G}(\eta,C)) \neq \emptyset$, *i.e., there exist* $h \in \mathcal{H}_0$ *and a metric* θ_h *on* K *such that* $(\theta_h, \Gamma(h,h)) \in \mathcal{G}(\eta,C)$.

Moreover, as in Sect. 6.2, a slight addition to our proof of Theorem 6.54 also shows the following theorem, which will be useful in studying further the problem of whether the infimum in (1.4) is attained for generalized Sierpiński carpets. Recall that $(\mathcal{F}/\mathbb{R}\mathbb{1}_K, \mathcal{E})$ is a Hilbert space as observed in Lemma 6.49.

Theorem 6.55 *(1)* $\widetilde{\mathcal{H}}_0(\mathcal{G}(\eta,C))$ *is compact in norm in* $(\mathcal{F}/\mathbb{R}\mathbb{1}_K, \mathcal{E})$.

(2) If $h \in \mathcal{H}_0(\mathcal{G}(\eta,C))$, *then* $\mathcal{E}(h \circ F_w, h \circ F_w)^{-1/2} h \circ F_w \in \mathcal{H}_0(\mathcal{G}(\eta,C))$ *for any* $w \in W_*$.

We remark that in Theorem 6.55-(2) we have $\mathcal{E}(h \circ F_w, h \circ F_w) = r_w \Gamma(h,h)(K_w) > 0$ for any $w \in W_*$ by Lemma 6.60 below and the lower inequality in (6.73) for $\mu = \Gamma(h,h)$.

The rest of this subsection is devoted to the proof of Theorems 6.54 and 6.55, which goes in exactly the same way as that of Theorem 6.16 and Proposition 6.17 except for some modifications required due to $\#V_0 = \infty = \dim \mathcal{H}_0$ and explained in detail below.

Proposition 6.56 $\mathcal{G}(\eta,C)$ *is a compact subset of* $C(K \times K) \times \mathcal{P}(K)$.

Proof The proof of Proposition 6.18 remains valid also in this case, except that $R_{\mathcal{E}}$ needs to be replaced by d and that s in the last paragraph needs to be defined as $s := l^{-|w|}$. □

Corollary 6.57 *Let* $\{(\theta_n, \mu_n)\}_{n \in \mathbb{N}} \subset \mathcal{G}(\eta,C)$, $\mu \in \mathcal{P}(K)$ *and suppose that* $\{\mu_n\}_{n \in \mathbb{N}}$ *converges to* μ *in* $\mathcal{P}(K)$. *Then there exists a metric* θ *on* K *such that* $(\theta, \mu) \in \mathcal{G}(\eta,C)$.

Proof The proof of Corollary 6.19 remains valid with Proposition 6.56 applied in place of Proposition 6.18. □

Lemma 6.58 *Let $(\theta, \mu) \in \mathcal{G}(\eta, C)$, $w \in W_*$ and define $(\theta_w, \mu_w) \in C(K \times K) \times \mathcal{P}(K)$ by (6.43). Then $(\theta_w, \mu_w) \in \mathcal{G}(\eta, C)$.*

Proof The proof of Lemma 6.20 remains valid also in this case, except that $R_{\mathcal{E}}$ needs to be replaced by d. $\square$

Lemma 6.59 *Let $w \in W_*$.*

(1) $\int_K |u \circ F_w|\, dm = (\#S)^{|w|} \int_{K_w} |u|\, dm$ for any Borel measurable function $u \colon K \to [-\infty, \infty]$, and hence a bounded linear operator from $L^2(K, m)$ to itself is defined by $u \mapsto u \circ F_w$.
(2) $u \circ F_w \in \mathcal{F}$ and (6.8) holds for any $u, v \in \mathcal{F}$.
(3) $h \circ F_w \in \mathcal{H}_0$ for any $h \in \mathcal{H}_0$.

Proof (1) is immediate from $m = (\#S)^{|w|} m \circ F_w$, and (2) follows from (6.7), (6.8), the denseness of $\mathcal{F} \cap C(K)$ in $(\mathcal{F}, \mathcal{E}_1)$ and the completeness of $(\mathcal{F}, \mathcal{E}_1)$; see [55, Proof of Lemma 3.3]. To see (3), let $h \in \mathcal{H}_0$, and let $v \in \mathcal{F} \cap C(K)$ satisfy $\mathrm{supp}_m[v] \subset K \setminus V_0$. Then since $K_\tau \cap K_w = F_\tau(V_0) \cap F_w(V_0)$ for any $\tau \in W_{|w|} \setminus \{w\}$ by [60, Proposition 1.3.5-(2)], we can define $v^w \in C(K)$ by $v^w|_{K_w} := v \circ F_w^{-1}$ and $v^w|_{K \setminus K_w} := 0$ and have $v^w \in \mathcal{F} \cap C(K)$ by (6.7) and $\mathrm{supp}_m[v^w] \subset K_w \setminus F_w(V_0) = K_w \setminus V_{|w|} \subset K \setminus V_0$, and it therefore follows from (6.8) for h, v^w and $h \in \mathcal{H}_0$ that $r_w^{-1}\mathcal{E}(h \circ F_w, v) = \sum_{\tau \in W_{|w|}} r_\tau^{-1}\mathcal{E}(h \circ F_\tau, v^w \circ F_\tau) = \mathcal{E}(h, v^w) = 0$, proving $h \circ F_w \in \mathcal{H}_0$. $\square$

Lemma 6.60 *Suppose that $\mathcal{G} \neq \emptyset$. Let $u \in \mathcal{F}$ and $w \in W_*$. Then $\Gamma(u, u)(F_w(A)) = r_w^{-1}\Gamma(u \circ F_w, u \circ F_w)(A)$ for any Borel subset A of K, and in particular $\Gamma(u, u)(K_w) = r_w^{-1}\mathcal{E}(u \circ F_w, u \circ F_w)$. Moreover, if $\Gamma(u, u)(K_w) > 0$, then (6.44) holds for any Borel subset A of K.*

Proof First, by Lemma 6.59-(2) and [45, Lemma 3.11-(ii)] it holds, *independently of the assumption $\mathcal{G} \neq \emptyset$*, that for any $v \in \mathcal{F}$ and any $n \in \mathbb{N} \cup \{0\}$,

$$\Gamma(v, v)(A) = \sum_{\tau \in W_n} \frac{1}{r_\tau} \Gamma(v \circ F_\tau, v \circ F_\tau)(F_\tau^{-1}(A)) \quad \text{for any Borel subset } A \text{ of } K. \tag{6.77}$$

By $\mathcal{G} \neq \emptyset$ we can take $(\theta, \mu) \in \mathcal{G}$, then μ is a minimal energy-dominant measure of $(\mathcal{E}, \mathcal{F})$ by Theorem 6.51 and Proposition 2.11-(b), thus $\Gamma(v, v) \ll \mu$ for any $v \in \mathcal{F}$, and hence

$$\Gamma(v, v)(V_0) = 0 \qquad \text{for any } v \in \mathcal{F} \tag{6.78}$$

since $\mu(V_0) = 0$ by Theorem 6.51. Recalling that for any $\tau \in W_{|w|} \setminus \{w\}$ we have $K_\tau \cap K_w = F_\tau(V_0) \cap F_w(V_0)$ by [60, Proposition 1.3.5-(2)] and hence

$F_\tau^{-1}(K_w) = F_\tau^{-1}(K_\tau \cap K_w) \subset V_0$, we see from (6.77) and (6.78) that for any Borel subset A of K,

$$\begin{aligned}\Gamma(u,u)(F_w(A)) &= \sum_{\tau \in W_{|w|}} \frac{1}{r_\tau}\Gamma(u \circ F_\tau, u \circ F_\tau)(F_\tau^{-1}(F_w(A))) \\ &= \frac{1}{r_w}\Gamma(u \circ F_w, u \circ F_w)(A).\end{aligned}$$

The rest of the proof goes in exactly the same way as that of Lemma 6.21. □

Remark 6.61 In fact, (6.78) holds without supposing $\mathcal{G} \neq \emptyset$ by [47, Proposition 4.15] and hence so does Lemma 6.60. The proof of (6.78) presented in [47, Section 5], however, is long and difficult, and since we later use Lemma 6.60 only under the supposition that $\mathcal{G} \neq \emptyset$, we have decided to suppose it explicitly to keep our present arguments independent of the demanding result [47, Proposition 4.15].

Lemma 6.62 *Suppose that $\mathcal{G} \neq \emptyset$, let $(\theta, \mu) \in \mathcal{G}$, $u \in \mathcal{F}$ and set $f := d\Gamma(u,u)/d\mu$. Then μ-a.e. $x \in K$ is a (d,μ)-*Lebesgue point *for f, i.e., satisfies*

$$\lim_{s \downarrow 0} \frac{1}{\mu(B_d(x,s))} \int_{B_d(x,s)} |f(y) - f(x)|\, d\mu(y) = 0. \tag{6.79}$$

Proof The proof of Lemma 6.22 remains valid also in this case, except that $R_\mathcal{E}$ needs to be replaced by d. □

Lemma 6.63 *Suppose that $\mathcal{G} \neq \emptyset$, let $(\theta, \mu) \in \mathcal{G}$, $u \in \mathcal{F}$, let $f : K \to [0,\infty)$ be a Borel measurable μ-version of $d\Gamma(u,u)/d\mu$ and let $x \in K$ satisfy (6.79). Then (6.47) holds for any $\omega \in \pi^{-1}(x)$ and any $w \in W_*$.*

Proof The proof of Lemma 6.23 remains valid also in this case, except that $R_\mathcal{E}$ needs to be replaced by d. □

Proposition 6.64 *Suppose that $\mathcal{G} \neq \emptyset$, let $(\theta, \mu) \in \mathcal{G}$, $u \in \mathcal{F}$, let $f : K \to [0,\infty)$ be a Borel measurable μ-version of $d\Gamma(u,u)/d\mu$, let $x \in K$ satisfy (6.79) and $f(x) > 0$, and let $\omega \in \pi^{-1}(x)$. For each $n \in \mathbb{N} \cup \{0\}$, define $\mu_n := \mu_{[\omega]_n} \in \mathcal{P}(K)$ by (6.43) with $w = [\omega]_n$ and, noting that $\Gamma(u,u)(K_{[\omega]_n}) > 0$ by Lemma 6.63, define $u_n := u_{[\omega]_n} \in \mathcal{F}$ by (6.44) with $w = [\omega]_n$. If $v \in \mathcal{F}$ and $\{n_k\}_{k \in \mathbb{N}} \subset \mathbb{N}$ is strictly increasing and satisfies $\lim_{k \to \infty} \mathcal{E}(v - u_{n_k}, v - u_{n_k}) = 0$, then $\Gamma(v,v) \in \mathcal{P}(K)$ and $\{\mu_{n_k}\}_{k \in \mathbb{N}}$ converges to $\Gamma(v,v)$ in $\mathcal{P}(K)$.*

Proof The proof of Proposition 6.24 remains valid also in this case, except that it is because of $\mathcal{G} \neq \emptyset$, (6.78) and Lemma 6.60 that $\Gamma(v,v)(F_w(V_0)) = 0$ for any $w \in W_*$. □

So far, except for the issue of the validity of (6.78) treated in the proof of Lemma 6.60 and Remark 6.61, our current discussion has been almost the same as the corresponding part of Sect. 6.2. On the other hand, the concluding parts of the proofs of Proposition 6.17-(1) and Theorem 6.16 rely on the compactness of $\{h \in \mathcal{H}_0/\mathbb{R}\mathbb{1}_K \mid \mathcal{E}(h,h) = 1\}$ implied by $\dim \mathcal{H}_0/\mathbb{R}\mathbb{1}_K < \infty$ and thereby cannot be extended directly to the present case of a generalized Sierpiński carpet, where $\dim \mathcal{H}_0/\mathbb{R}\mathbb{1}_K = \infty$ by $\#V_0 = \infty$. We overcome this difficulty by establishing Proposition 6.67 below on the basis of Lemma 6.49 and the following lemma and applying it with the help of the compactness of $\mathcal{G}(\eta, C)$ from Proposition 6.56.

Lemma 6.65 (Reverse Poincaré inequality) *There exists $C_{\mathrm{RP}} \in (0,\infty)$ such that for any $(x,s) \in K \times (0,\infty)$, any $a \in \mathbb{R}$ and any function $h \in \mathcal{F}$ that is $\mathcal{E}$-harmonic on $B_d(x,2s)$,*

$$\int_{B_d(x,s)} d\Gamma(h,h) \le \frac{C_{\mathrm{RP}}}{s^{d_{\mathrm{w}}}} \int_{B_d(x,2s)\setminus B_d(x,s)} |h-a|^2\, dm. \tag{6.80}$$

Proof This is a special case of [56, Lemma 3.3] with $\Psi(s) = s^{d_{\mathrm{w}}}$, whose assumption $\mathrm{CS}(\Psi)$ formulated in [56, Definition 2.6-(b)] is implied in the current situation by Lemma 6.43, Theorem 6.48 and [1, Theorem 5.5]; see also Remark 6.66 below. □

Remark 6.66 To be precise, [56, Lemma 3.3] assumes additionally that $h \in L^\infty(X,m)$, but this assumption can be dropped by replacing [56, the first four lines of (3.9)] with the following, where $h_n := (-n) \vee (h \wedge n)$ for $n \in \mathbb{N}$:

$$\begin{aligned}
0 &= \lim_{n\to\infty} \mathcal{E}(h, h_n\varphi^2) = \lim_{n\to\infty} \Gamma(h, h_n\varphi^2)(X) \qquad \text{(by [56, (3.7) and (2.4)])}\\
&= \lim_{n\to\infty}\left(\int_X \varphi^2\, d\Gamma(h,h_n) + 2\int_X \varphi h_n\, d\Gamma(h,\varphi)\right) \quad \text{(by [33, Lemma 3.2.5])}\\
&\ge \limsup_{n\to\infty}\left(\int_X \varphi^2\, d\Gamma(h,h_n) - 2\sqrt{\int_X \varphi^2\, d\Gamma(h,h)\int_X h_n^2\, d\Gamma(\varphi,\varphi)}\right)\\
&\qquad \text{(by [33, Proof of Lemma 5.6.1])}\\
&= \int_X \varphi^2\, d\Gamma(h,h) - 2\sqrt{\int_X \varphi^2\, d\Gamma(h,h)\int_X h^2\, d\Gamma(\varphi,\varphi)}\\
&\qquad \text{(by the Cauchy–Schwarz inequality for } \textstyle\int_X \varphi^2\, d\Gamma(\cdot,\cdot)\\
&\qquad \text{and [33, Theorem 1.4.2-(iii)]).}
\end{aligned}$$

Proposition 6.67 *Let $\{h_k\}_{k\in\mathbb{N}} \subset \mathcal{H}_0$ converge weakly in $(\mathcal{F}, \mathcal{E}_1)$ to $h \in \mathcal{F}$(,* so that $h \in \mathcal{H}_0$ since $\mathcal{H}_0$ is weakly closed in $(\mathcal{F}, \mathcal{E}_1)$)*, and assume that there exist a Radon measure ν on K and $n_0 \in \mathbb{N}$ such that $\nu(F_w(V_0)) = 0$ and*

$\limsup_{k\to\infty} r_w^{-1}\mathcal{E}(h_k \circ F_w, h_k \circ F_w) \leq \nu(K_w)$ *for any* $w \in W_*$ *satisfying* $|w| \geq n_0$ *and* $K_w \cap V_0 \neq \emptyset$. *Then* $\lim_{k\to\infty} \mathcal{E}_1(h - h_k, h - h_k) = 0$.

Proof Thanks to the weak convergence in $(\mathcal{F}, \mathcal{E}_1)$ of $\{h_k\}_{k\in\mathbb{N}}$ to h, we have

$$\lim_{k\to\infty} \int_K |h - h_k|^2 \, dm = 0 \tag{6.81}$$

by the compactness of the inclusion map from $(\mathcal{F}, \mathcal{E}_1)$ to $L^2(K, m)$ stated in Lemma 6.49 and [71, Chapter 21, Theorem 9], and for any $w \in W_*$, $\{h_k \circ F_w + \mathbb{R}\mathbb{1}_K\}_{k\in\mathbb{N}} \subset \mathcal{H}_0/\mathbb{R}\mathbb{1}_K$ converges weakly in $(\mathcal{F}/\mathbb{R}\mathbb{1}_K, \mathcal{E})$ to $h \circ F_w + \mathbb{R}\mathbb{1}_K$ since $u \mapsto u \circ F_w + \mathbb{R}\mathbb{1}_K$ is a bounded linear operator from $(\mathcal{F}, \mathcal{E}_1)$ to $(\mathcal{F}/\mathbb{R}\mathbb{1}_K, \mathcal{E})$ by Lemma 6.59-(2). In particular,

$$\mathcal{E}(h \circ F_w, h \circ F_w) \leq \liminf_{k\to\infty} \mathcal{E}(h_k \circ F_w, h_k \circ F_w) \qquad \text{for any } w \in W_*. \tag{6.82}$$

Moreover, recalling that $K \setminus V_0$ is open in K and non-empty and letting $\tau \in W_*$, $z_\tau \in K_\tau$ and $s_\tau \in (0, \infty)$ satisfy $K_\tau \subset B_d(z_\tau, s_\tau) \subset B_d(z_\tau, 2s_\tau) \subset K \setminus V_0$, we see from (6.77) with $A = K_\tau$ and Lemma 6.65 with $a = 0$ that for any $v \in \mathcal{H}_0$,

$$\frac{1}{r_\tau}\mathcal{E}(v \circ F_\tau, v \circ F_\tau) \leq \Gamma(v, v)(K_\tau) \leq \Gamma(v, v)(B_d(z_\tau, s_\tau)) \leq \frac{C_{\mathrm{RP}}}{s_\tau^{d_{\mathrm{w}}}} \int_K v^2 \, dm, \tag{6.83}$$

which with $v = h - h_k$ for $k \in \mathbb{N}$, together with (6.81), shows that

$$\lim_{k\to\infty} \mathcal{E}((h - h_k) \circ F_\tau, (h - h_k) \circ F_\tau) = 0. \tag{6.84}$$

Now let $n \in \mathbb{N}$ satisfy $n \geq n_0$, set $V_{0,n} := \bigcup_{w\in W_n,\, K_w\cap V_0\neq\emptyset} K_w$ and $W_n^\circ := \{\tau \in W_{n+N} \mid K_\tau \not\subset V_{0,n}\}$, so that $\{w \in W_n \mid K_w \cap V_0 \neq \emptyset\} \cup W_n^\circ$ is a partition of Σ (recall Definition 6.12-(2)) and each $\tau \in W_n^\circ$ satisfies (6.84) since $K_\tau \subset B_d(z_\tau, l^{-n-1}) \subset B_d(z_\tau, 2l^{-n-1}) \subset K \setminus V_0$ for any $z_\tau \in K_\tau$. Then since $\nu(F_w(V_0)) = 0$ and $\limsup_{k\to\infty} r_w^{-1}\mathcal{E}(h_k \circ F_w, h_k \circ F_w) \leq \nu(K_w)$ for any $w \in W_n$ with $K_w \cap V_0 \neq \emptyset$ and hence $\sum_{w\in W_n,\, K_w\cap V_0\neq\emptyset} \nu(K_w) = \nu(V_{0,n})$ and $\nu(V_0) = \nu\bigl(V_0 \cap \bigcup_{w\in W_n,\, K_w\cap V_0\neq\emptyset} F_w(V_0)\bigr) = 0$ by [60, Proposition 1.3.5-(2)] and $V_0 \subset V_n$, it follows from Lemma 6.59-(2), (6.84) for $\tau \in W_n^\circ$ and (6.82) that

$$
\begin{aligned}
&\limsup_{k\to\infty} \mathcal{E}(h-h_k, h-h_k) \\
&\quad = \limsup_{k\to\infty} \sum_{\tau\in\{w\in W_n \mid K_w\cap V_0\neq\emptyset\}\cup W_n^{\circ}} \frac{1}{r_\tau}\mathcal{E}((h-h_k)\circ F_\tau, (h-h_k)\circ F_\tau) \\
&\quad = \limsup_{k\to\infty} \sum_{w\in W_n,\, K_w\cap V_0\neq\emptyset} \frac{1}{r_w}\mathcal{E}(h\circ F_w - h_k\circ F_w, h\circ F_w - h_k\circ F_w) \\
&\quad \leq \limsup_{k\to\infty} \sum_{w\in W_n,\, K_w\cap V_0\neq\emptyset} \frac{2}{r_w}\bigl(\mathcal{E}(h\circ F_w, h\circ F_w) + \mathcal{E}(h_k\circ F_w, h_k\circ F_w)\bigr) \\
&\quad \leq \sum_{w\in W_n,\, K_w\cap V_0\neq\emptyset} \frac{2}{r_w}\Bigl(\mathcal{E}(h\circ F_w, h\circ F_w) + \limsup_{k\to\infty}\mathcal{E}(h_k\circ F_w, h_k\circ F_w)\Bigr) \\
&\quad \leq \sum_{w\in W_n,\, K_w\cap V_0\neq\emptyset} \frac{4}{r_w}\limsup_{k\to\infty}\mathcal{E}(h_k\circ F_w, h_k\circ F_w) \\
&\quad \leq \sum_{w\in W_n,\, K_w\cap V_0\neq\emptyset} 4\nu(K_w) = 4\nu(V_{0,n}) \xrightarrow{n\to\infty} 4\nu(V_0) = 0,
\end{aligned}
$$

which along with (6.81) proves $\lim_{k\to\infty}\mathcal{E}_1(h-h_k, h-h_k) = 0$. □

Proof of Theorem 6.55 Recall (6.73) for $\mathcal{G}(\eta, C)$, (6.33) for $\mathcal{H}_0(\mathcal{Z})$ and (6.34) for $\widetilde{\mathcal{H}}_0(\mathcal{Z})$.

(2) The proof of Proposition 6.17-(2) remains valid also in this case, except that Proposition 6.6-(2), Lemmas 6.21 and 6.20 need to be replaced by Lemma 6.59-(3), Lemmas 6.60 and 6.58 , respectively.

(1) Let $\{h_n\}_{n\in\mathbb{N}} \subset \mathcal{H}_0(\mathcal{G}(\eta, C))$, so that $\{\Gamma(h_n, h_n)\}_{n\in\mathbb{N}} \subset \mathcal{P}(K)$ and hence $\{h_n\}_{n\in\mathbb{N}} \subset \{h \in \mathcal{H}_0 \mid (\Gamma(h,h)(K) =)\, \mathcal{E}(h,h) = 1\}$. Setting $v_n := h_n - \int_K h_n\, dm \in \mathcal{H}_0$ for each $n \in \mathbb{N}$, by $\mathcal{E}(\mathbb{1}_K, \mathbb{1}_K) = 0$ and Lemma 6.49 we have $\mathcal{E}_1(v_n, v_n) \leq C_{\mathrm{P}} + 1$ for any $n \in \mathbb{N}$, therefore by [71, Chapter 10, Theorem 7] there exist $h \in \mathcal{F}$ and a strictly increasing sequence $\{n'_j\}_{j\in\mathbb{N}} \subset \mathbb{N}$ such that $\{v_{n'_j}\}_{j\in\mathbb{N}}$ converges weakly in $(\mathcal{F}, \mathcal{E}_1)$ to h, and $h \in \mathcal{H}_0$ since $\mathcal{H}_0$ is weakly closed in $(\mathcal{F}, \mathcal{E}_1)$. Further, recalling that $\mathcal{P}(K)$ is a compact metrizable topological space by [87, Theorems 9.1.5 and 9.1.9], we can choose a strictly increasing sequence $\{j_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ such that $\{\Gamma(h_{n_k}, h_{n_k})\}_{k\in\mathbb{N}}$ converges to some $\nu \in \mathcal{P}(K)$ in $\mathcal{P}(K)$, where $n_k := n'_{j_k}$, and then $(\vartheta, \nu) \in \mathcal{G}(\eta, C)$ for some metric ϑ on K by $\{h_{n_k}\}_{k\in\mathbb{N}} \subset \mathcal{H}_0(\mathcal{G}(\eta, C))$ and Corollary 6.57. In particular, for any $w \in W_*$ we have $\nu(F_w(V_0)) = 0$ by (6.73) and Theorem 6.51 and

$$
\limsup_{k\to\infty} \frac{1}{r_w}\mathcal{E}(v_{n_k}\circ F_w, v_{n_k}\circ F_w) \leq \limsup_{k\to\infty} \Gamma(h_{n_k}, h_{n_k})(K_w) \leq \nu(K_w) \tag{6.85}
$$

by $\mathcal{E}(\mathbb{1}_K, \mathbb{1}_K) = 0$ and (6.77) with $A = K_w$, thus $\lim_{k\to\infty} \mathcal{E}_1(h - v_{n_k}, h - v_{n_k}) = 0$ by Proposition 6.67 and hence $\lim_{k\to\infty} \mathcal{E}(h - h_{n_k}, h - h_{n_k}) = 0$ by $\mathcal{E}(\mathbb{1}_K, \mathbb{1}_K) = 0$. It follows that (6.45) holds for any Borel subset A of K, so that $\Gamma(h, h) \in \mathcal{P}(K)$ and $\{\Gamma(h_{n_k}, h_{n_k})\}_{k\in\mathbb{N}}$ converges to $\Gamma(h, h)$ in $\mathcal{P}(K)$, whence $(\vartheta, \Gamma(h, h)) = (\vartheta, \nu) \in \mathcal{G}(\eta, C)$. Therefore $h \in \mathcal{H}_0(\mathcal{G}(\eta, C))$, which together with $\lim_{k\to\infty} \mathcal{E}(h - h_{n_k}, h - h_{n_k}) = 0$ proves that $\widetilde{\mathcal{H}}_0(\mathcal{G}(\eta, C))$ is (sequentially) compact in norm in $(\mathcal{F}/\mathbb{R}\mathbb{1}_K, \mathcal{E})$. $\square$

Proof of Theorem 6.54 By the assumption $\mathcal{G}(\eta, C) \neq \emptyset$ we can take $(\theta, \mu) \in \mathcal{G}(\eta, C)$. Choose $u \in \mathcal{H}_0$ so that $\mathcal{E}(u, u) > 0$; such u exists, e.g., by [55, Propositions 3.7 and 3.10]. Let $f : K \to [0, \infty)$ be a Borel measurable μ-version of $d\Gamma(u, u)/d\mu$. Then $\mu\bigl(f^{-1}((0, \infty))\bigr) > 0$ by $\int_K f\, d\mu = \Gamma(u, u)(K) = \mathcal{E}(u, u) > 0$, and therefore by Lemma 6.62 there exists $x \in K$ with the properties (6.79) and $f(x) > 0$. Let $\omega \in \pi^{-1}(x)$, and for each $n \in \mathbb{N} \cup \{0\}$, as in Proposition 6.64 define $(\theta_n, \mu_n) := (\theta_{[\omega]_n}, \mu_{[\omega]_n}) \in C(K \times K) \times \mathcal{P}(K)$ by (6.43) and $u_n := u_{[\omega]_n} \in \mathcal{F}$ by (6.44) with $w = [\omega]_n$, so that $\{(\theta_n, \mu_n)\}_{n\in\mathbb{N}\cup\{0\}} \subset \mathcal{G}(\eta, C)$ by Lemma 6.58 and $\{u_n\}_{n\in\mathbb{N}\cup\{0\}} \subset \{h \in \mathcal{H}_0 \mid \mathcal{E}(h, h) = 1\}$ by Lemma 6.59-(3). Then setting $v_n := u_n - \int_K u_n\, dm \in \mathcal{H}_0$ for each $n \in \mathbb{N} \cup \{0\}$, by $\mathcal{E}(\mathbb{1}_K, \mathbb{1}_K) = 0$ and Lemma 6.49 we have $\mathcal{E}_1(v_n, v_n) \leq C_{\mathrm{P}} + 1$ for any $n \in \mathbb{N} \cup \{0\}$, hence by [71, Chapter 10, Theorem 7] there exist $h \in \mathcal{F}$ and a strictly increasing sequence $\{n'_j\}_{j\in\mathbb{N}} \subset \mathbb{N}$ such that $\{v_{n'_j}\}_{j\in\mathbb{N}}$ converges weakly in $(\mathcal{F}, \mathcal{E}_1)$ to h, and $h \in \mathcal{H}_0$ since $\mathcal{H}_0$ is weakly closed in $(\mathcal{F}, \mathcal{E}_1)$. Further, by $\{(\theta_{n'_j}, \mu_{n'_j})\}_{j\in\mathbb{N}} \subset \mathcal{G}(\eta, C)$, Proposition 6.56 and the metrizability of $C(K \times K) \times \mathcal{P}(K)$ we can choose a strictly increasing sequence $\{j_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ such that $\{(\theta_{n_k}, \mu_{n_k})\}_{k\in\mathbb{N}}$ converges to some $(\vartheta, \nu) \in \mathcal{G}(\eta, C)$ in $C(K \times K) \times \mathcal{P}(K)$, where $n_k := n'_{j_k}$. Then for any $w \in W_*$, we have $\nu(F_w(V_0)) = 0$ by (6.73) and Theorem 6.51, and by using $\mathcal{E}(\mathbb{1}_K, \mathbb{1}_K) = 0$, (6.77) with $A = K_w$, (6.44) from Lemma 6.60, the definition of $\mu_{n_k} = \mu_{[\omega]_{n_k}}$ from (6.43), (6.47) from Lemma 6.63, $f(x) \in (0, \infty)$ and the convergence of $\{\mu_{n_k}\}_{k\in\mathbb{N}}$ to ν in $\mathcal{P}(K)$, we obtain

$$
\begin{aligned}
&\limsup_{k\to\infty} \frac{1}{r_w} \mathcal{E}(v_{n_k} \circ F_w, v_{n_k} \circ F_w) \\
&\leq \limsup_{k\to\infty} \Gamma(u_{n_k}, u_{n_k})(K_w) = \limsup_{k\to\infty} \frac{\Gamma(u, u)(K_{[\omega]_{n_k} w})}{\Gamma(u, u)(K_{[\omega]_{n_k}})} \\
&= \limsup_{k\to\infty} \frac{\Gamma(u, u)(K_{[\omega]_{n_k} w})/\mu(K_{[\omega]_{n_k} w})}{\Gamma(u, u)(K_{[\omega]_{n_k}})/\mu(K_{[\omega]_{n_k}})} \mu_{n_k}(K_w) \\
&= \frac{f(x)}{f(x)} \limsup_{k\to\infty} \mu_{n_k}(K_w) \leq \nu(K_w).
\end{aligned}
\tag{6.86}
$$

Thus $\lim_{k\to\infty} \mathcal{E}_1(h - v_{n_k}, h - v_{n_k}) = 0$ by Proposition 6.67, hence $\lim_{k\to\infty} \mathcal{E}(h - u_{n_k}, h - u_{n_k}) = 0$ by $\mathcal{E}(\mathbb{1}_K, \mathbb{1}_K) = 0$, and it follows from Proposition 6.64 that $\{\mu_{n_k}\}_{k\in\mathbb{N}}$ converges to $\Gamma(h, h) \in \mathcal{P}(K)$ in $\mathcal{P}(K)$, whence $(\vartheta, \Gamma(h, h)) = (\vartheta, \nu) \in \mathcal{G}(\eta, C)$. □

Remark 6.68 Note that the relatively short proof of Proposition 6.67 above is enabled by the assumed properties of the Radon measure ν on K. The existence of such ν seems difficult to verify for a general sequence $\{h_k\}_{k\in\mathbb{N}} \subset \mathcal{H}_0$ converging weakly in $(\mathcal{F}, \mathcal{E}_1)$, but can be obtained via the compactness of $\mathcal{G}(\eta, C)$ from Proposition 6.56 in the situations of the proofs of Theorems 6.55 and 6.54 above as observed in (6.85) and (6.86). In fact, Hino [47, Proposition 4.18] has proved a similar sequential compactness *without assuming the existence of such* ν, at the price of its long difficult proof given in [47, Section 5].

7 Further remarks and open problems

We conclude the present paper with mentioning some related open problems.

Problem 7.1 Does $d_{\mathrm{cw}} < \infty$ characterize the elliptic Harnack inequality for symmetric jump process?

It is not clear if the equivalence between (a) and (b) in Theorem 2.10 extends to jump processes. Despite the progress made in the diffusion case, the characterization and stability of the elliptic Harnack inequality is still open for jump processes.

Our study of the Gaussian uniformization problem in Sect. 5 gives only partial answers, both for a general MMD space and for the MMD space of Brownian motion on $\mathbb{R}^n$, except for an explicit answer for $\mathbb{R}$ in Theorem 5.18 and an implicit (unsatisfactory) one for $\mathbb{R}^2$ in Proposition 5.16. In particular, the following problem is left open.

Problem 7.2 Characterize explicitly all the Gaussian admissible measures for the MMD space of Brownian motion on $\mathbb{R}^n$, $n \geq 2$ (see (5.3) for the definition).

For Problem 7.3 below, let $(K, R_{\mathcal{E}}, m, \mathcal{E}, \mathcal{F})$ denote the MMD space resulting as in Sect. 6.1 from a post-critically finite self-similar structure $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$ with $\#S \geq 2$ and K connected and a regular harmonic structure $(D, \mathbf{r})$ on $\mathcal{L}$. Recall that $\mathcal{H}_0$ has been defined in Definition 6.5 as $\mathcal{H}_0 := \{h \in C(K) \mid h \text{ is 0-harmonic}\}$ under this setting.

Problem 7.3 Provide simple sufficient conditions for the non-attainment of the conformal walk dimension for the MMD space $(K, R_{\mathcal{E}}, m, \mathcal{E}, \mathcal{F})$.

In view of the non-attainment results in Sect. 6.3 for the Vicsek set (Corollary 6.30) and the higher-dimensional Sierpiński gaskets (Theorem 6.35), it seems natural to expect that the conformal walk dimension would typically fail to be attained for the MMD space $(K, R_{\mathcal{E}}, m, \mathcal{E}, \mathcal{F})$ as in Problem 7.3. This expectation, however, does not seem very easy to verify, since the behavior of the linear maps $\mathcal{H}_0 \ni h \mapsto h \circ F_w \in \mathcal{H}_0$ could be difficult to analyze for a given pair of $\mathcal{L}$ and $(D, \mathbf{r})$ and might not allow a proof by contradiction based on Theorem 6.16 and Proposition 6.17 as achieved in the proof of Theorem 6.35.

For Problems 7.4, 7.5 and 7.6 below, let $N, K, d, m, V_0, (\mathcal{E}, \mathcal{F}), \mathcal{H}_0$ be as introduced in Sect. 6.4 as pieces of the framework of the canonical self-similar Dirichlet form on an arbitrary generalized Sierpiński carpet K; in particular, recall that $\mathcal{H}_0$ has been defined in Definition 6.50 as $\mathcal{H}_0 := \{h \in \mathcal{F} \mid h \text{ is } \mathcal{E}\text{-harmonic on } K \setminus V_0 = K \cap (0, 1)^N\}$.

Problem 7.4 Is the conformal walk dimension attained for the canonical MMD space $(K, d, m, \mathcal{E}, \mathcal{F})$ over the generalized Sierpiński carpet K?

As a matter of fact, the authors have proved in a recent ongoing work that *the conformal walk dimension is NOT attained for the two-dimensional standard Sierpiński carpet*. It is therefore likely that the answer to Problem 7.4 would be negative also for a given generalized Sierpiński carpet. Note, however, that in view of Theorem 6.54 the negative answer to Problem 7.4 would mean only that the energy measure $\Gamma(h, h)$ of every $h \in \mathcal{H}_0 \setminus \mathbb{R}\mathbb{1}_K$ would fail to satisfy $\Gamma(h, h) \in \mathcal{G}(K, d, m, \mathcal{E}, \mathcal{F})$, which is a much stronger requirement than just VD of $(K, d, \Gamma(h, h))$. In particular, the following problem remains non-trivial regardless of the actual answer to Problem 7.4.

Problem 7.5 Does there exist $h \in \mathcal{H}_0 \setminus \mathbb{R}\mathbb{1}_K$ such that its energy measure $\Gamma(h, h)$ satisfies the volume doubling property with respect to the Euclidean metric d?

Problem 7.5 appears very challenging since we do not know even the answer to the following much simpler question.

Problem 7.6 Does there exist $h \in \mathcal{H}_0 \setminus \mathbb{R}\mathbb{1}_K$ such that its energy measure $\Gamma(h, h)$ has full support?

It is tempting to conjecture that the energy measure $\Gamma(h, h)$ of every $h \in \mathcal{H}_0 \setminus \mathbb{R}\mathbb{1}_K$ has full support. This can be viewed as a unique continuation principle for harmonic functions on a given generalized Sierpiński carpet.

We expect that there is a version of Theorems 6.16 and 6.54 for Ahlfors regular conformal dimension on self-similar spaces. In particular, we expect that if the Ahlfors regular conformal dimension $p > 1$ is attained on a self-similar space then there exists a "p-harmonic function" such that its "energy measure" is a p-Ahlfors regular measure with respect to a metric in the conformal gauge. This motivates the following problem.

Problem 7.7 Define non-linear analogs of Dirichlet space, energy measure and harmonic functions on self-similar spaces (for example, the Sierpiński carpet).

Using the theory of Dirichlet forms and its relationship to diffusion process we have a notion of Sobolev space $W^{1,2}$ on the Sierpiński carpet whose seminorm can be formally thought of as the integral $(\int |\nabla f|^2)^{1/2}$ on a dense subspace of L^2. For example in [70], the $W^{1,2}$-seminorm and the Dirichlet energy $\int |\nabla f|^2$ is constructed as suitably renormailized version of discrete Dirichlet energy $\sum_{x\sim y}(\hat{f}(x) - \hat{f}(y))^2$, where $\hat{f}$ is a discretization of the function f on a sequence of graph approximations V_n. A key ingredient in the constructions in [4] and [70] is the following submultiplicative and supermultiplicative inequalities for resistance between opposite faces. Let R_n denote the resistance between the opposite faces for the nth level approximation of the Sierpiński carpet. Then the submultiplicative and supermultiplicative inequalities are given by $R_n R_m \gtrsim R_{m+n}$ and $R_n R_m \lesssim R_{m+n}$. These inequalities have been generalized in the non-linear context for the Sierpiński carpet by Bourdon and Kleiner for any $p > 1$ [21, Lemma 4.4]. This suggests that one could construct a non-linear version of Dirichlet form with a $W^{1,p}$ seminorm (formally denoted by $\|\nabla f\|_p$). This seminorm is defined on a dense subspace $\mathcal{F}_p$ of L^p and is conjectured to satisfy the following properties:

1. (Closability) If f_n is a Cauchy sequence in the $\|\nabla f\|_p$-seminorm and if $f_n \to 0$ in L^p then f_n converges to 0 in the $\|\nabla f\|_p$-seminorm.
2. (Regularity) Let K denote the Sierpiński carpet. Then $\mathcal{F}_p \cap C(K)$ is dense with respect to the uniform norm in $C(K)$ and is dense with respect to the $f \mapsto \|\nabla f\|_p + \|f\|_p$ norm.

This yields the notion of p-harmonic functions which are defined as minimizers of the p-Dirichlet energy $\|\nabla f\|_p^p$. On the basis of Theorem 6.54, we have the following conjecture: if the Sierpiński carpet attains the Ahlfors regular conformal dimension, then there exists a p-harmonic function whose energy measure (formally written as the measure $A \mapsto \int_A |\nabla f|^p$) is an optimal Ahlfors regular measure, where p is the Ahlfors regular conformal dimension of the Sierpiński carpet. The previous discussion on unique continuation question in the linear case ($p = 2$) also applies to p-harmonic functions due to the above mentioned relationship to the attainment problem for the Ahlfors regular conformal dimension. These conjectures serve as a motivation to develop a theory of non-linear Dirichlet forms on fractals and develop methods to obtain the elliptic Harnack inequality and quantitative unique continuation principle for p-harmonic functions. Added in revision: there has been recent progress on Problem 7.7 in [68,85]. These results are quite satisfactory when p is strictly larger than the Ahlfors regular conformal dimension.

Acknowledgements The authors would like to thank Ryosuke Shimizu for his valuable comments on an earlier version of this paper, and the anonymous referees for their careful reading of this manuscript and helpful suggestions. The authors would also like to express gratitude to Shouhei Honda for having communicated to the first-named author the idea of taking scaling limits at Lebesgue points to reduce the analysis to the case of harmonic functions, which the authors have adapted in Sects. 6.2 and 6.4. This paper was revised while the second-named author was at the Mathematical Sciences Research Institute in Berkeley, California, during the Spring 2022 semester on Analysis and Geometry of Random Spaces, which was supported by the National Science Foundation under Grant No. DMS-1928930.

Appendix: index of words, phrases, symbols and abbreviations

Words and phrases

- $(1, p)$-Poincaré inequality—Definition 5.3
- attainment problem—Problem 1.3, second paragraph of Sect. 5
- A_∞-related—Definition 5.5
- bi-Lipschitz, bi-Lipschitz equivalent—Definition 3.1
- boundary of a hyperbolic space—fourth paragraph of Sect. 3.1
- caloric—Definition 2.4
- canonical Dirichlet form on $\mathrm{GSC}(N, l, S)$—Definition 6.46
- conformal gauge—Definition 1.1
- critical set—Definition 6.3
- distortion function—Definition 3.1
- doubling (measure)—Definition 3.17
- doubling (metric space)—first paragraph of Sect. 3.2
- $\mathcal{E}$-harmonic—Definition 2.4
- energy measure—Definition 2.1
- full quasi-support—Definition 2.8
- Gaussian admissible measures—(5.2), (5.3)
- Gaussian uniformization problem—Problem 1.3, second paragraph of Sect. 5
- generalized Sierpiński carpet—Definition 6.39
- gentle—Definition 3.15
- Gromov hyperbolic space—second paragraph of Sect. 3.1

Symbols

- $\mathbb{1}_A = \mathbb{1}_A^X$: indicator function of $A \subset X$ on a set X—Notation 1.4-(f)
- $\mathcal{A}(X, d, m, \mathcal{E}, \mathcal{F})$: admissible measures—Definition 2.8
- $B(x, r) = B_d(x, r)$: open ball in metric d—first paragraph of Sect. 2.1
- $\overline{B}(x, r) = \overline{B}_d(x, r)$: closed ball in metric d—first paragraph of Sect. 2.1
- $c_{R_{\mathcal{E}}}$—Lemma 6.11
- $C(X)$: space of continuous functions on X—Notation 1.4-(g)
- $C_c(X)$: space of continuous functions on X with compact supports—Notation 1.4-(g)
- $\mathrm{Cap}_1(A)$: 1-capacity of a set A—(2.6)
- $\mathrm{Cap}(A, B)$: capacity between sets A and B—(2.7)
- $\mathcal{C}_{\mathcal{L}}$: critical set of $\mathcal{L}$—Definition 6.3
- d_{cw}: conformal walk dimension—Definition 1.2
- d_{f}: Euclidean Hausdorff dimension of $\mathrm{GSC}(N, l, S)$—Framework 6.38
- d_h: maximal semi-metric induced by h—Definition 5.12
- d_{H}: Hausdorff dimension of $(K, R_{\mathcal{E}})$—Lemma 6.8 and preceding paragraph
- $\mathrm{diam}_d(A) = \mathrm{diam}(A, d)$: diameter of A in metric d—first paragraph of Sect. 2.1
- d_{int}: intrinsic metric—Definition 2.3
- d_{int}^{μ}: μ-intrinsic metric—Definition 6.25
- d_{w}: walk dimension of the Dirichlet form on $\mathrm{GSC}(N, l, S)$—Definition 6.46
- $(D, \mathbf{r})$: harmonic structure—Definition 6.4
- $\mathcal{D}_l(B)$: descendants of generation l—(3.12)
- $D_{\mathcal{S}}$: combinatorial metric on the hyperbolic filling—Definition 3.6.
- $\mathcal{E}_1$: inner product on $\mathcal{F}$ for a Dirichlet form $(\mathcal{E}, \mathcal{F})$—second paragraph of Sect. 2.1
- $\mathcal{E}^{(n)}$: discrete Dirichlet form on V_n—(6.2), (6.3)
- F_w—Definition 6.1
- $\mathcal{F}_e$: extended Dirichlet space—Definition 2.4
- $\mathcal{F}_{\mathrm{loc}}$: space of functions locally in $\mathcal{F}$—Definition 2.3
- $\mathrm{GSC}(N, l, S)$—Framework 6.38
- $\mathcal{G}(X, d, m, \mathcal{E}, \mathcal{F})$: Gaussian admissible measures—(5.2), (5.3)
- $\mathcal{G}_\beta(X, d, m, \mathcal{E}, \mathcal{F})$: sub-Gaussian admissible measures—(5.1)
- $\mathcal{G} = \mathcal{G}_{\mathcal{L},(D,\mathbf{r})}$ (for post-critically finite $\mathcal{L}$)—Definition 6.15
- $\mathcal{G} = \mathcal{G}_{N,l,S}$ (for $\mathrm{GSC}(N, l, S)$)—Definition 6.53
- $\mathcal{G}(\eta, C) = \mathcal{G}_{\mathcal{L},(D,\mathbf{r})}(\eta, C)$ (for post-critically finite $\mathcal{L}$)—Definition 6.15
- $\mathcal{G}(\eta, C) = \mathcal{G}_{N,l,S}(\eta, C)$ (for $\mathrm{GSC}(N, l, S)$)—Definition 6.53
- Homeo^+: group of homeomorphisms of $[0, \infty)$—Definition 6.15
- $\mathcal{H}_0$ (for $\mathrm{GSC}(N, l, S)$)—Definition 6.50

- $\mathcal{H}_0(\mathcal{Z})$: set of 0-harmonic functions attaining d_{cw}—(6.33) in Definition 6.15
- $\widetilde{\mathcal{H}}_0(\mathcal{Z})$—(6.34) in Definition 6.15
- $\mathcal{H}_n$ (for post-critically finite $\mathcal{L}$)—Definition 6.5
- $\mathrm{int}_{\mathbb{R}^N}(A)$: interior of $A \subset \mathbb{R}^N$ in $\mathbb{R}^N$—paragraph before Definition 6.39
- $\mathcal{I}_0$: group of cubic symmetries of $\mathrm{GSC}(N, l, S)$—Definition 6.44
- $\mathcal{J}(X, d)$: conformal gauge of (X, d)—Definition 1.1
- $K_s(x)$—Definition 6.13
- K_w—Definition 6.1
- $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$: self-similar structure—Definition 6.2
- $\mathrm{Lip}\, u(x)$: pointwise Lipschitz constant—Definition 5.1
- $\mathrm{Lip}(X)$: space of Lipschitz functions on X—Definition 5.1
- $N^{1,p}(X) = N^{1,p}(X, d, m)$—Definition 5.2
- $N^{1,p}_{\mathrm{loc}}(X) = N^{1,p}_{\mathrm{loc}}(X, d, m)$—Definition 5.2
- $\mathbb{N}$: set of *positive* integers—Notation 1.4-(c)
- $\mathcal{N}$: non-peripheral vertices—Definition 3.23
- $\mathcal{P}(K)$: set of Borel probability measures on K—(6.30) in Definition 6.15
- $\mathcal{P}_{\mathcal{L}}$: post-critical set of $\mathcal{L}$—Definition 6.3
- Q_0: N-dimensional unit cube $[0, 1]^N$—Framework 6.38
- Q_1—Framework 6.38
- r_w (for post-critically finite $\mathcal{L}$)—(6.3)
- r_w (for $\mathrm{GSC}(N, l, S)$)—Theorem 6.51
- $R_{\mathcal{E}}$: resistance metric—paragraph following Definition 6.4, especially (RF4)
- $\mathrm{supp}_m[f]$: support of $|f|\,dm$—second paragraph of Sect. 2.1
- $\mathcal{S}$: hyperbolic filling—Definition 3.6, Sect. 3.2
- $\mathscr{S} = \{\Lambda_s\}_{s\in(0,1]}$: scale on Σ associated with $\mathbf{r} = (r_i)_{i\in S}$—Definition 6.13
- u_w—(6.44) in Lemma 6.21
- $U_s(x)$—Definition 6.13
- V_n: set of level-n boundary points of $\mathcal{L}$—Definition 6.3
- V_*: set of arbitrary level boundary points of $\mathcal{L}$—Definition 6.3
- w^∞—Definition 6.1
- W_n: set of words of length n—Definition 6.1
- W_*: set of words of arbitrary length—Definition 6.1
- $(X, d, m, \mathcal{E}, \mathcal{F})$: MMD (metric measure Dirichlet) space—Sect. 2.1
- $\Gamma_{k+1}(B)$—Definition 3.22
- $\Gamma(f, f)$: energy measure—Definition 2.1
- $\tilde{\eta}$: multiplicative dual of η—paragraphs before Theorems 6.16 and 6.54
- θ_w (for $(\theta, \mu) \in \mathcal{G}(\eta, C)$)—(6.43) in Lemma 6.20
- Λ_s—Definition 6.13
- $\Lambda_{s,x}$—Definition 6.13
- $\Lambda^1_{s,x}$—Definition 6.13
- μ_w (for $(\theta, \mu) \in \mathcal{G}(\eta, C)$)—(6.43) in Lemma 6.20

- π (for a self-similar structure $\mathcal{L}$): projection from Σ to K—Definition 6.2
- σ—Definition 6.1
- σ_w—Definition 6.1
- Σ: (one-sided) shift space—Definition 6.1
- Σ_w—Definition 6.1
- $[\omega]_n$—Definition 6.1
- $(\cdot)^+$: positive part of a $[-\infty, \infty]$-valued quantity—Notation 1.4-(e)
- $(\cdot)^-$: negative part of a $[-\infty, \infty]$-valued quantity—Notation 1.4-(e)
- $(\cdot|\cdot)_\cdot$: Gromov product—(3.1)
- $\#A$: cardinality (number of elements) of a set A—Notation 1.4-(d)
- $\leq$ (for measures)—Notation 1.4-(h)
- $\leq$ (for elements of W_*)—Definition 6.12
- $\leq$ (for partitions of Σ): refinement—Definition 6.12
- $\lesssim$: inequality up to constant multiples—Notation 1.4-(b)
- $\ll$: absolute continuity of a measure with respect to another—Notation 1.4-(h)
- $\vee$: maximum of two $[-\infty, \infty]$-valued quantities—Notation 1.4-(e)
- $\wedge$: minimum of two $[-\infty, \infty]$-valued quantities—Notation 1.4-(e)

Abbreviations

- cap(β): capacity estimate—Definition 4.2
- CS(β): cutoff Sobolev inequality—Definition 4.3
- (E): enhanced subadditivity estimate—Definition 3.23
- EHI: elliptic Harnack inequality—Definition 2.5
- (GSC1),(GSC2),(GSC3),(GSC4): requirements for GSC(N, l, S) to be a generalized Sierpiński carpet—Definition 6.39
- (GSCDF)—Theorem 6.45
- (H1),(H2),(H3): conditions on weight functions in a hyperbolic filling—Assumption 3.11
- HKE(β): heat kernel estimate—Definition 4.1
- $(\mathrm{ND})_{\mathbb{N}}$,$(\mathrm{ND})_2$,(NDF): equivalent formulations of (GSC3)—Proposition 6.40
- PHI(β): parabolic Harnack inequality—Definition 2.5
- PI(β): Poincaré inequality—Definition 4.3
- RVD: reverse volume doubling property—Definition 3.17
- (S1),(S2): conditions on weight functions in a hyperbolic filling—Theorem 3.24
- VD: volume doubling property—Definition 3.17

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