70. Use the partial fraction decomposition

$$\frac{1}{x^3+x^2+x} = \frac{A}{x} + \frac{Bx+C}{x^2+x+1}$$
$$= \frac{A(x^2+x+1)+Bx^2+Cx}{x^3+x^2+x}$$
$$\Rightarrow \begin{cases} A+B=0 \\ A+C=0 \\ A=1 \end{cases} \Rightarrow A=1,\ B=-1,\ C=-1.$$

Therefore,

$$\int \frac{dx}{x^3+x^2+x}$$
$$= \int \frac{dx}{x} - \int \frac{x+1}{x^2+x+1}\,dx \quad \text{Let } u = x+\tfrac{1}{2}$$
$$du = dx$$
$$= \ln|x| - \int \frac{u+\frac{1}{2}}{u^2+\frac{3}{4}}\,du$$
$$= \ln|x| - \frac{1}{2}\ln\!\left(x^2+x+1\right) - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C.$$

8. Let $I = \int_a^b f(x)\,dx$, and the interval $[a, b]$ be subdivided into $2n$ subintervals of equal length $h = (b-a)/2n$. Let $y_j = f(x_j)$ and $x_j = a + jh$ for $0 \le j \le 2n$, then

$$\begin{aligned} S_{2n} &= \frac{1}{3}\left(\frac{b-a}{2n}\right)\Big[y_0 + 4y_1 + 2y_2 + \cdots \\ &\qquad + 2y_{2n-2} + 4y_{2n-1} + y_{2n}\Big] \\ &= \frac{1}{3}\left(\frac{b-a}{2n}\right)\left[y_0 + 4\sum_{j=1}^{2n-1} y_j - 2\sum_{j=1}^{n-1} y_{2j} + y_{2n}\right] \end{aligned}$$

and

$$T_{2n} = \frac{1}{2}\left(\frac{b-a}{2n}\right)\left(y_0 + 2\sum_{j=1}^{2n-1} y_j + y_{2n}\right)$$

$$T_n = \frac{1}{2}\left(\frac{b-a}{n}\right)\left(y_0 + 2\sum_{j=1}^{n-1} y_{2j} + y_{2n}\right).$$

Since $T_{2n} = \frac{1}{2}(T_n + M_n) \Rightarrow M_n = 2T_{2n} - T_n$, then

$$\frac{T_n + 2M_n}{3} = \frac{T_n + 2(2T_{2n} - T_n)}{3} = \frac{4T_{2n} - T_n}{3}$$

$$\frac{2T_{2n} + M_n}{3} = \frac{2T_{2n} + 2T_{2n} - T_n}{3} = \frac{4T_{2n} - T_n}{3}.$$

Hence,

$$\frac{T_n + 2M_n}{3} = \frac{2T_{2n} + M_n}{3} = \frac{4T_{2n} - T_n}{3}.$$

Using the formulas of T_{2n} and T_n obtained above,

$$\begin{aligned} &\frac{4T_{2n} - T_n}{3} \\ &= \frac{1}{3}\left[\frac{4}{2}\left(\frac{b-a}{2n}\right)\left(y_0 + 2\sum_{j=1}^{2n-1} y_j + y_{2n}\right)\right. \\ &\qquad \left. - \frac{1}{2}\left(\frac{b-a}{n}\right)\left(y_0 + 2\sum_{j=1}^{n-1} y_{2j} + y_{2n}\right)\right] \\ &= \frac{1}{3}\left(\frac{b-a}{2n}\right)\left[y_0 + 4\sum_{j=1}^{2n-1} y_j - 2\sum_{j=1}^{n-1} y_{2j} + y_{2n}\right] \\ &= S_{2n}. \end{aligned}$$

Hence,

$$S_{2n} = \frac{4T_{2n} - T_n}{3} = \frac{T_n + 2M_n}{3} = \frac{2T_{2n} + M_n}{3}.$$

44. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^{2n-1}}{(2n-1)!} = \sin x$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n}\pi^{2n-1}}{(2n-1)!} = -\sin\pi = 0$$

$$\sum_{n=2}^{\infty} \frac{(-1)^{n}\pi^{2n-4}}{(2n-1)!} = \frac{1}{\pi^3}\left(0 - \frac{(-1)\pi}{1!}\right) = \frac{1}{\pi^2}.$$

46. $\displaystyle \lim_{x\to 0} \frac{(x-\tan^{-1}x)(e^{2x}-1)}{2x^2-1+cos(2x)}$

$$= \lim_{x\to 0} \frac{\left(x - x + \dfrac{x^3}{3} - \dfrac{x^5}{5} + \cdots\right)\left(2x + \dfrac{4x^2}{2!} + \cdots\right)}{2x^2 - 1 + 1 - \dfrac{4x^2}{2!} + \dfrac{16x^4}{4!} - \cdots}$$

$$= \lim_{x\to 0} \frac{x^4\left(\dfrac{2}{3} + \cdots\right)}{x^4\left(\dfrac{2}{3} + \cdots\right)} = 1.$$

7. Let $f(x) = \sum_{k=0}^{\infty} a_k x^{2k+1}$, where $a_k = \dfrac{2^{2k}k!}{(2k+1)!}$.

a) Since

$$\begin{aligned}
\lim_{k\to\infty} \left| \frac{a_{k+1}x^{2k+3}}{a_k x^{2k+1}} \right| \\
&= |x|^2 \lim_{k\to\infty} \frac{2^{2k+2}}{2^{2k}} \cdot \frac{(k+1)!}{k!} \cdot \frac{(2k+1)!}{(2k+3)!} \\
&= |x|^2 \lim_{k\to\infty} \frac{4k+4}{(2k+3)(2k+2)} = 0
\end{aligned}$$

for all x, the series for $f(x)$ converges for all x. Its radius of convergence is infinite.

b) $f'(x) = \sum_{k=0}^{\infty} \dfrac{2^{2k}k!}{(2k+1)!}(2k+1)x^{2k} = 1 + \sum_{k=1}^{\infty} \dfrac{2^{2k}k!}{(2k)!}x^{2k}$

$$1 + 2xf(x) = 1 + \sum_{k=0}^{\infty} \frac{2^{2k+1}k!}{(2k+1)!}x^{2k+2}$$

(replace k with $k-1$)

$$= 1 + \sum_{k=1}^{\infty} \frac{2^{2k-1}(k-1)!}{(2k-1)!}x^{2k}$$

$$= 1 + \sum_{k=1}^{\infty} \frac{2^{2k}k!}{(2k)!}x^{2k} = f'(x).$$

c) $\dfrac{d}{dx}\left(e^{-x^2} f(x)\right) = e^{-x^2}\left(f'(x) - 2xf(x)\right) = e^{-x^2}$.

d) Since $f(0) = 0$, we have

$$e^{-x^2} f(x) - f(0) = \int_0^x \frac{d}{dt}\left(e^{-t^2} f(t)\right) dt = \int_0^x e^{-t^2}\, dt$$

$$f(x) = e^{x^2} \int_0^x e^{-t^2}\, dt.$$

5. $$V = \int_0^6 (2+z)(8-z)\,dz = \int_0^6 (16+6z-z^2)\,dz$$
$$= \left(16z + 3z^2 - \frac{z^3}{3}\right)\Big|_0^6 = 132 \text{ ft}^3$$

12.
$$\begin{aligned}
s &= \int_{\pi/6}^{\pi/4} \sqrt{1+\tan^2 x}\, dx \\
&= \int_{\pi/6}^{\pi/4} \sec x\, dx = \ln|\sec x + \tan x|\Big|_{\pi/6}^{\pi/4} \\
&= \ln(\sqrt{2}+1) - \ln\left(\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right) \\
&= \ln \frac{\sqrt{2}+1}{\sqrt{3}} \text{ units.}
\end{aligned}$$

4. The height of each triangular face is $2\sqrt{3}$ m and the height of the pyramid is $2\sqrt{2}$ m. Let the angle between the triangular face and the base be θ, then $\sin\theta = \sqrt{\frac{2}{3}}$ and $\cos\theta = \frac{1}{\sqrt{3}}$.

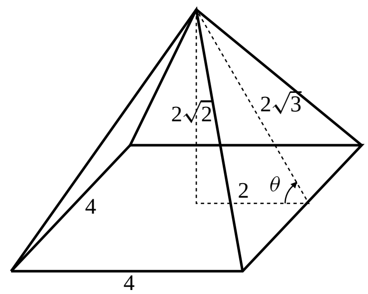

Fig. 6-4

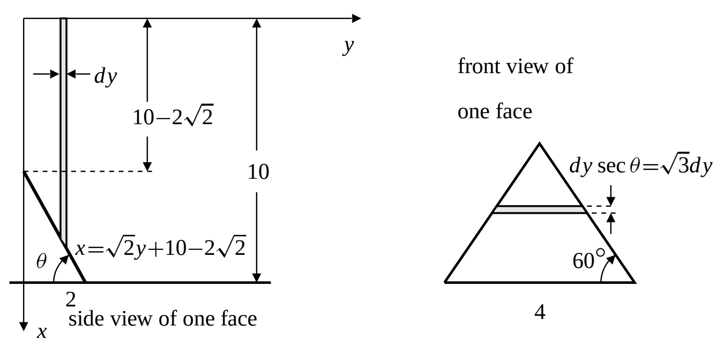

Fig. 6-4

A vertical slice of water with thickness dy at a distance y from the vertex of the pyramid exerts a force on the shaded strip shown in the front view, which has area $2\sqrt{3}y\,dy$ m^2 and which is at depth $\sqrt{2}y + 10 - 2\sqrt{2}$ m. Hence, the force exerted on the triangular face is

$$
\begin{aligned}
F &= \rho g \int_0^2 (\sqrt{2}y + 10 - 2\sqrt{2})2\sqrt{3}y\,dy \\
&= 2\sqrt{3}(9800)\left[\frac{\sqrt{2}}{3}y^3 + (5 - \sqrt{2})y^2\right]\Big|_0^2 \\
&\approx 6.1495 \times 10^5 \text{ N.}
\end{aligned}
$$

4. Area $= \dfrac{1}{2}\displaystyle\int_0^{\pi/3} \sin^2 3\theta \, d\theta = \dfrac{1}{4}\displaystyle\int_0^{\pi/3} (1 - \cos 6\theta)\, d\theta$

$$= \frac{1}{4}\left(\theta - \frac{1}{6}\sin 6\theta\right)\Bigg|_0^{\pi/3} = \frac{\pi}{12} \text{ sq. units.}$$

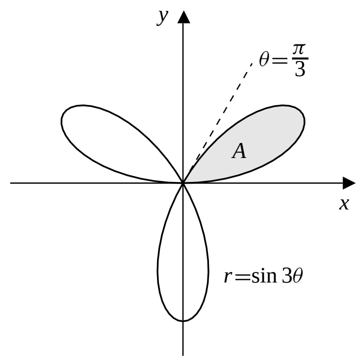

Fig. 6-4

27. $$a_n = \frac{(n!)^2}{(2n)!} = \frac{(1 \cdot 2 \cdot 3 \cdots n)(1 \cdot 2 \cdot 3 \cdots n)}{1 \cdot 2 \cdot 3 \cdots n \cdot (n+1) \cdot (n+2) \cdots 2n}$$
$$= \frac{1}{n+1} \cdot \frac{2}{n+2} \cdot \frac{3}{n+3} \cdots \frac{n}{n+n} \leq \left(\frac{1}{2}\right)^n .$$
Thus $\lim a_n = 0$.

36. a) "If $\lim a_n = \infty$ and $\lim b_n = L > 0$, then $\lim a_n\, b_n = \infty$" is TRUE. Let R be an arbitrary, large positive number. Since $\lim a_n = \infty$, and $L > 0$, it must be true that $a_n \geq \dfrac{2R}{L}$ for n sufficiently large. Since $\lim b_n = L$, it must also be that $b_n \geq \dfrac{L}{2}$ for n sufficiently large. Therefore $a_n b_n \geq \dfrac{2R}{L}\dfrac{L}{2} = R$ for n sufficiently large. Since R is arbitrary, $\lim a_n b_n = \infty$.

b) "If $\lim a_n = \infty$ and $\lim b_n = -\infty$, then $\lim(a_n + b_n) = 0$" is FALSE. Let $a_n = 1 + n$ and $b_n = -n$; then $\lim a_n = \infty$ and $\lim b_n = -\infty$ but $\lim(a_n + b_n) = 1$.

c) "If $\lim a_n = \infty$ and $\lim b_n = -\infty$, then $\lim a_n\, b_n = -\infty$" is TRUE. Let R be an arbitrary, large positive number. Since $\lim a_n = \infty$ and $\lim b_n = -\infty$, we must have $a_n \geq \sqrt{R}$ and $b_n \leq -\sqrt{R}$, for all sufficiently large n. Thus $a_n b_n \leq -R$, and $\lim a_n b_n = -\infty$.

d) "If neither $\{a_n\}$ nor $\{b_n\}$ converges, then $\{a_n\, b_n\}$ does not converge" is FALSE. Let $a_n = b_n = (-1)^n$; then $\lim a_n$ and $\lim b_n$ both diverge. But $a_n\, b_n = (-1)^{2n} = 1$ and $\{a_n\, b_n\}$ does converge (to 1).

e) "If $\{|a_n|\}$ converges, then $\{a_n\}$ converges" is FALSE. Let $a_n = (-1)^n$. Then $\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} 1 = 1$, but $\lim_{n\to\infty} a_n$ does not exist.

12. We have $\frac{dy}{dx} + \frac{2y}{x} = \frac{1}{x^2}$. Let $\mu = \int \frac{2}{x}\,dx = 2\ln x = \ln x^2$, then $e^{\mu} = x^2$, and

$$\begin{aligned}
&\frac{d}{dx}(x^2y) = x^2\frac{dy}{dx} + 2xy \\
&= x^2\left(\frac{dy}{dx} + \frac{2y}{x}\right) = x^2\left(\frac{1}{x^2}\right) = 1 \\
&\Rightarrow \quad x^2y = \int dx = x + C \\
&\Rightarrow \quad y = \frac{1}{x} + \frac{C}{x^2}.
\end{aligned}$$

1. The expected winnings on a toss of the coin are

$$\$1 \times 0.49 + \$2 \times 0.49 + \$50 \times 0.02 = \$2.47.$$

If you pay this much to play one game, in the long term you can expect to break even.

41. $\displaystyle\int_0^\infty \frac{dx}{xe^x} = \left(\int_0^1 + \int_1^\infty\right) \frac{dx}{xe^x}$. But

$$\int_0^1 \frac{dx}{xe^x} \geq \frac{1}{e} \int_0^1 \frac{dx}{x} = \infty.$$

Thus the given integral must diverge to infinity.