7. For $\sum_{n=0}^{\infty} \dfrac{1+5^n}{n!}\, x^n$ we have $R = \lim \dfrac{1+5^n}{n!} \cdot \dfrac{(n+1)!}{1+5^{n+1}} = \infty$. The radius of convergence is infinite; the centre of convergence is 0; the interval of convergence is the whole real line $(-\infty, \infty)$.

8. We have $\sum_{n=1}^{\infty} \frac{(4x-1)^n}{n^n} = \sum_{n=1}^{\infty} \left(\frac{4}{n}\right)^n \left(x - \frac{1}{4}\right)^n$. The centre of convergence is $x = \frac{1}{4}$. The radius of convergence is

$$\begin{aligned} R &= \lim \frac{4^n}{n^n} \cdot \frac{(n+1)^{n+1}}{4^{n+1}} \\ &= \frac{1}{4} \lim \left(\frac{n+1}{n}\right)^n (n+1) = \infty. \end{aligned}$$

Hence, the interval of convergence is $(-\infty, \infty)$.

17. Let $x+2=t$, so $x=t-2$. Then

$$\begin{aligned}\frac{1}{x^2}=\frac{1}{(2-t)^2}&=\sum_{n=0}^{\infty}\frac{(n+1)t^n}{2^{n+2}}\\&=\sum_{n=0}^{\infty}\frac{(n+1)(x+2)^n}{2^{n+2}},\quad(-4<x<0).\end{aligned}$$

19. We have

$$\begin{aligned}\frac{x^3}{1-2x^2} &= x^3\left(\sum_{n=0}^{\infty}(2x^2)^n\right)\\ &= \sum_{n=0}^{\infty}2^n x^{2n+3}, \qquad \left(-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}\right).\end{aligned}$$

20. Let $t = x + 1$. Then $x = t - 1$, and

$$\begin{aligned} e^{2x+3} &= e^{2t+1} = e\,e^{2t} \\ &= e \sum_{n=0}^{\infty} \frac{2^n t^n}{n!} \qquad \text{(for all } t\text{)} \\ &= \sum_{n=0}^{\infty} \frac{e 2^n (x+1)^n}{n!} \qquad \text{(for all } x\text{).} \end{aligned}$$

21. Let $t = x - (\pi/4)$, so $x = t + (\pi/4)$. Then

$$\begin{aligned} f(x) &= \sin x - \cos x \\ &= \sin\left(t + \frac{\pi}{4}\right) - \cos\left(t + \frac{\pi}{4}\right) \\ &= \frac{1}{\sqrt{2}}\big[(\sin t + \cos t) - (\cos t - \sin t)\big] \\ &= \sqrt{2}\sin t = \sqrt{2}\sum_{n=0}^{\infty}(-1)^n \frac{t^{2n+1}}{(2n+1)!} \\ &= \sqrt{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(x - \frac{\pi}{4}\right)^{2n+1} \qquad \text{(for all } x\text{)}. \end{aligned}$$

29. From Example 7, $\sum_{n=1}^{\infty} n^2 x^{n-1} = \frac{1+x}{(1-x)^3}$ for $-1 < x < 1$. Putting $x = 1/\pi$, we get

$$\sum_{n=0}^{\infty} \frac{(n+1)^2}{\pi^n} = \sum_{k=1}^{\infty} \frac{k^2}{\pi^{k-1}} = \frac{1+\frac{1}{\pi}}{(1-\frac{1}{\pi})^3} = \frac{\pi^2(\pi+1)}{(\pi-1)^3}.$$

30. From Example 5(a),

$$\sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}, \quad (-1 < x < 1).$$

Differentiate with respect to x and then replace n by $n+1$:

$$\sum_{n=2}^{\infty} n(n-1)x^{n-2} = \frac{2}{(1-x)^3}, \quad (-1 < x < 1)$$

$$\sum_{n=1}^{\infty} (n+1)nx^{n-1} = \frac{2}{(1-x)^3}, \quad (-1 < x < 1).$$

Now let $x = -1/2$:

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n(n+1)}{2^{n-1}} = \frac{16}{27}.$$

Finally, multiply by $-1/2$:

$$\sum_{n=1}^{\infty} (-1)^{n} \frac{n(n+1)}{2^{n}} = -\frac{8}{27}.$$

31. Since $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} = \ln(1+x)$ for $-1 < x \leq 1$, therefore

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n2^n} = \ln\left(1+\frac{1}{2}\right) = \ln\frac{3}{2}.$$

34. $$x^3 - \frac{x^9}{3! \times 4} + \frac{x^{15}}{5! \times 16} - \frac{x^{21}}{7! \times 64} + \frac{x^{27}}{9! \times 256} - \cdots$$
$$= 2\left[\frac{x^3}{2} - \frac{1}{3!}\left(\frac{x^3}{2}\right)^3 + \frac{1}{5!}\left(\frac{x^3}{2}\right)^5 - \cdots\right]$$
$$= 2\sin\left(\frac{x^3}{2}\right) \qquad \text{(for all } x\text{)}.$$

35. $1 + \frac{x^2}{3!} + \frac{x^4}{5!} + \frac{x^6}{7!} + \cdots$

$$= \frac{1}{x} \sinh x = \frac{e^x - e^{-x}}{2x}$$

if $x \neq 0$. The sum is 1 if $x = 0$.

36. $$1 + \frac{1}{2 \times 2!} + \frac{1}{4 \times 3!} + \frac{1}{8 \times 4!} + \cdots$$
$$= 2\left[\frac{1}{2} + \frac{1}{2!}\left(\frac{1}{2}\right)^2 + \frac{1}{3!}\left(\frac{1}{2}\right)^3 + \cdots\right]$$
$$= 2\left(e^{1/2} - 1\right).$$

42. We want to prove that $f(x) = P_n(x) + E_n(x)$, where P_n is the nth-order Taylor polynomial for f about c and

$$E_n(x) = \frac{1}{n!}\int_c^x (x-t)^n f^{(n+1)}(t)\,dt.$$

(a) The Fundamental Theorem of Calculus written in the form

$$f(x) = f(c) + \int_c^x f'(t)\,dt = P_0(x) + E_0(x)$$

is the case $n = 0$ of the above formula. We now apply integration by parts to the integral, setting

$$\begin{aligned} U &= f'(t), & dV &= dt, \\ dU &= f''(t)\,dt, & V &= -(x-t). \end{aligned}$$

(We have broken our usual rule about not including a constant of integration with V. In this case we have included the constant $-x$ in V in order to have V vanish when $t = x$.) We have

$$\begin{aligned} f(x) &= f(c) - f'(t)(x-t)\Big|_{t=c}^{t=x} + \int_c^x (x-t)f''(t)\,dt \\ &= f(c) + f'(c)(x-c) + \int_c^x (x-t)f''(t)\,dt \\ &= P_1(x) + E_1(x). \end{aligned}$$

We have now proved the case $n = 1$ of the formula.

(b) We complete the proof for general n by mathematical induction. Suppose the formula holds for some $n = k$:

$$\begin{aligned} f(x) &= P_k(x) + E_k(x) \\ &= P_k(x) + \frac{1}{k!}\int_c^x (x-t)^k f^{(k+1)}(t)\,dt. \end{aligned}$$

Again we integrate by parts. Let

$$\begin{aligned} U &= f^{(k+1)}(t), & dV &= (x-t)^k\,dt, \\ dU &= f^{(k+2)}(t)\,dt, & V &= \frac{-1}{k+1}(x-t)^{k+1}. \end{aligned}$$

We have

$$\begin{aligned} f(x) &= P_k(x) + \frac{1}{k!}\left(-\frac{f^{(k+1)}(t)(x-t)^{k+1}}{k+1}\Bigg|_{t=c}^{t=x} \right. \\ &\qquad \left. + \int_c^x \frac{(x-t)^{k+1} f^{(k+2)}(t)}{k+1}\,dt\right) \\ &= P_k(x) + \frac{f^{(k+1)}(c)}{(k+1)!}(x-c)^{k+1} \\ &\qquad + \frac{1}{(k+1)!}\int_c^x (x-t)^{k+1} f^{(k+2)}(t)\,dt \end{aligned}$$

43. If $f(x) = \ln(1+x)$, then

$$f'(x) = \frac{1}{1+x}, \quad f''(x) = \frac{-1}{(1+x)^2}, \quad f'''(x) = \frac{2}{(1+x)^3},$$
$$f^{(4)}(x) = \frac{-3!}{(1+x)^4}, \quad \ldots, \quad f^{(n)} = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$$

and

$$f(0) = 0, \quad f'(0) = 1, \quad f''(0) = -1, \quad f'''(0) = 2,$$
$$f^{(4)}(0) = -3!, \quad \ldots, \quad f^{(n)}(0) = (-1)^{n-1}(n-1)!.$$

Therefore, the Taylor Formula is

$$f(x) = x + \frac{-1}{2!}x^2 + \frac{2}{3!}x^3 + \frac{-3!}{4!}x^4 + \cdots + \frac{(-1)^{n-1}(n-1)!}{n!}x^n + E_n(x)$$

where

$$\begin{aligned} E_n(x) &= \frac{1}{n!}\int_0^x (x-t)^n f^{(n+1)}(t)\,dt \\ &= \frac{1}{n!}\int_0^x (x-t)^n \frac{(-1)^n n!}{(1+t)^{n+1}}\,dt \\ &= (-1)^n \int_0^x \frac{(x-t)^n}{(1+t)^{n+1}}\,dt. \end{aligned}$$

If $0 \le t \le x \le 1$, then $1+t \ge 1$ and

$$|E_n(x)| \le \int_0^x (x-t)^n\,dt = \frac{x^{n+1}}{n+1} \le \frac{1}{n+1} \to 0$$

as $n \to \infty$.
If $-1 < x \le t \le 0$, then

$$\left|\frac{x-t}{1+t}\right| = \frac{t-x}{1+t} \le |x|,$$

because $\dfrac{t-x}{1+t}$ increases from 0 to $-x = |x|$ as t increases from x to 0. Thus,

$$|E_n(x)| < \frac{1}{1+x}\int_0^{|x|} |x|^n\,dt = \frac{|x|^{n+1}}{1+x} \to 0$$

as $n \to \infty$ since $|x| < 1$. Therefore,

$$f(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n},$$

for $-1 < x \le 1$.