7. $x = \cosh t$, $y = \sinh^2 t$.
Parabola $x^2 - y = 1$, or $y = x^2 - 1$, traversed left to right.

10. $x = \cos t + \sin t$, $y = \cos t - \sin t$, $(0 \le t \le 2\pi)$
The circle $x^2 + y^2 = 2$, traversed clockwise, starting and ending at $(1, 1)$.

11. $x = \dfrac{4}{1+t^2} \qquad y = t^3 - 3t$

$\dfrac{dx}{dt} = -\dfrac{8t}{(1+t^2)^2} \qquad \dfrac{dy}{dt} = 3(t^2-1)$

Horizontal tangent at $t = \pm 1$, i.e., at $(2, \pm 2)$.
Vertical tangent at $t = 0$, i.e., at $(4, 0)$.
Self-intersection at $t = \pm\sqrt{3}$, i.e., at $(1, 0)$.

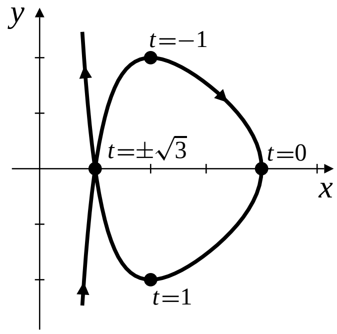

Fig. R-11

14. $x = t^3 - 3t \qquad y = t^3 - 12t$

$$\frac{dx}{dt} = 3(t^2 - 1) \qquad \frac{dy}{dt} = 3(t^2 - 4)$$

Horizontal tangent at $t = \pm 2$, i.e., at $(2, -16)$ and $(-2, 16)$.
Vertical tangent at $t = \pm 1$, i.e., at $(2, 11)$ and $(-2, -11)$.

$$\text{Slope } \frac{dy}{dx} = \frac{t^2 - 4}{t^2 - 1} \quad \begin{cases} > 0 & \text{if } |t| > 2 \text{ or } |t| < 1 \\ < 0 & \text{if } 1 < |t| < 2 \end{cases}$$

Slope $\to 1$ as $t \to \pm\infty$.

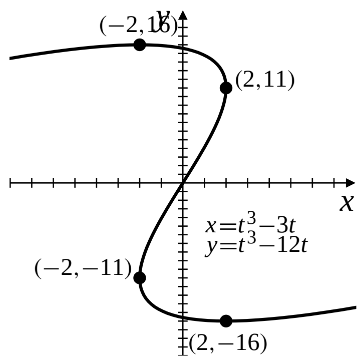

Fig. R-14

17. $x = e^t - t$, $y = 4e^{t/2}$, $(0 \le t \le 2)$. Length is

$$\begin{aligned} L &= \int_0^2 \sqrt{(e^t - 1)^2 + 4e^t}\,dt \\ &= \int_0^2 \sqrt{(e^t + 1)^2}\,dt = \int_0^2 (e^t + 1)\,dt \\ &= (e^t + t)\Big|_0^2 = e^2 + 1 \text{ units.} \end{aligned}$$

23. $r = 1 + 2\cos(2\theta)$

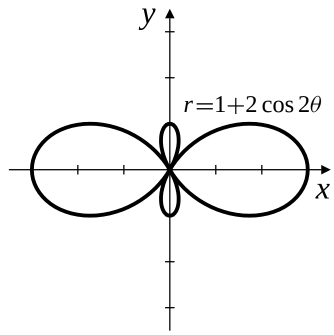

Fig. R-23

26. Area of a small loop:

$$\begin{aligned} A &= 2 \times \frac{1}{2} \int_{\pi/3}^{\pi/2} (1 + 2\cos(2\theta))^2 \, d\theta \\ &= \int_{\pi/3}^{\pi/2} [1 + 4\cos(2\theta) + 2(1 + \cos(4\theta))] \, d\theta \\ &= \left(3\theta + 2\sin(2\theta) + \frac{1}{2}\sin(4\theta) \right) \Bigg|_{\pi/3}^{\pi/2} \\ &= \frac{\pi}{2} - \frac{3\sqrt{3}}{4} \text{ sq. units.} \end{aligned}$$

28. $r\cos\theta = x = 1/4$ and $r = 1+\cos\theta$ intersect where

$$\begin{aligned}
1+\cos\theta &= \frac{1}{4\cos\theta}\\
4\cos^2\theta + 4\cos\theta - 1 &= 0\\
\cos\theta &= \frac{-4\pm\sqrt{16+16}}{8} = \frac{\pm\sqrt{2}-1}{2}.
\end{aligned}$$

Only $(\sqrt{2}-1)/2$ is between -1 and 1, so is a possible value of $\cos\theta$. Let $\theta_0 = \cos^{-1}\dfrac{\sqrt{2}-1}{2}$. Then

$$\sin\theta_0 = \sqrt{1-\left(\frac{\sqrt{2}-1}{2}\right)^2} = \frac{\sqrt{1+2\sqrt{2}}}{2}.$$

By symmetry, the area inside $r = 1+\cos\theta$ to the left of the line $x = 1/4$ is

$$\begin{aligned}
A &= 2\times\frac{1}{2}\int_{\theta_0}^{\pi}\left(1+2\cos\theta+\frac{1+\cos(2\theta)}{2}\right)d\theta + \cos\theta_0\sin\theta_0\\
&= \frac{3}{2}(\pi-\theta_0) + \left(2\sin\theta+\frac{1}{4}\sin(2\theta)\right)\Bigg|_{\theta_0}^{\pi}\\
&\quad + \frac{(\sqrt{2}-1)\sqrt{1+2\sqrt{2}}}{4}\\
&= \frac{3}{2}\left(\pi-\cos^{-1}\frac{\sqrt{2}-1}{2}\right) + \sqrt{1+2\sqrt{2}}\left(\frac{\sqrt{2}-9}{8}\right) \text{ sq. units.}
\end{aligned}$$

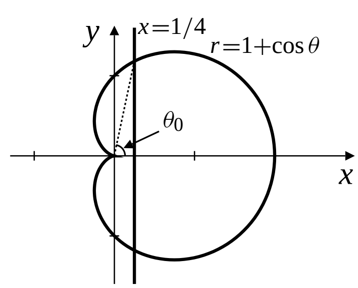

Fig. R-28

2. Let S_1 and S_2 be two spheres inscribed in the cylinder, one on each side of the plane that intersects the cylinder in the curve C that we are trying to show is an ellipse. Let the spheres be tangent to the cylinder around the circles C_1 and C_2, and suppose they are also tangent to the plane at the points F_1 and F_2, respectively, as shown in the figure.

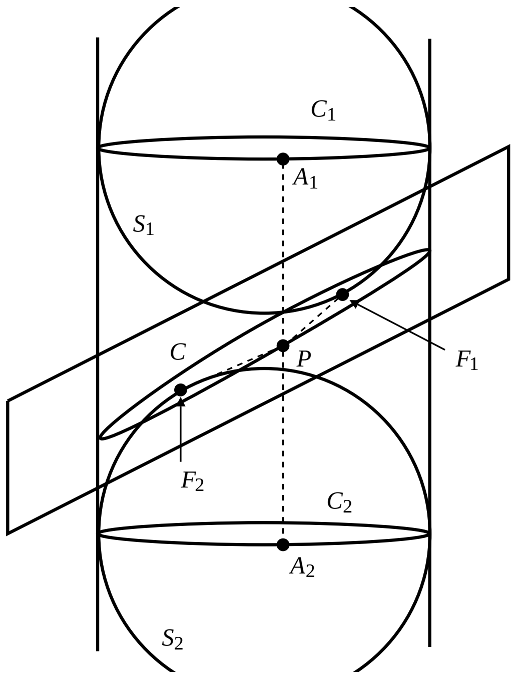

Fig. C-2

Let P be any point on C. Let A_1A_2 be the line through P that lies on the cylinder, with A_1 on C_1 and A_2 on C_2. Then $PF_1 = PA_1$ because both lengths are of tangents drawn to the sphere S_1 from the same exterior point P. Similarly, $PF_2 = PA_2$. Hence

$$PF_1 + PF_2 = PA_1 + PA_2 = A_1A_2,$$

which is constant, the distance between the centres of the two spheres. Thus C must be an ellipse, with foci at F_1 and F_2.

18. The two curves $r^2 = 2\sin 2\theta$ and $r = 2\cos\theta$ intersect where

$$\begin{aligned} & 2\sin 2\theta = 4\cos^2\theta \\ & 4\sin\theta\cos\theta = 4\cos^2\theta \\ & (\sin\theta - \cos\theta)\cos\theta = 0 \\ \Leftrightarrow\quad & \sin\theta = \cos\theta \text{ or } \cos\theta = 0, \end{aligned}$$

i.e., at $P_1 = \left[\sqrt{2}, \frac{\pi}{4}\right]$ and $P_2 = (0, 0)$.

For $r^2 = 2\sin 2\theta$ we have $2r\frac{dr}{d\theta} = 4\cos 2\theta$. At P_1 we have $r = \sqrt{2}$ and $dr/d\theta = 0$. Thus the angle ψ between the curve and the radial line $\theta = \pi/4$ is $\psi = \pi/2$.

For $r = 2\cos\theta$ we have $dr/d\theta = -2\sin\theta$, so the angle between this curve and the radial line $\theta = \pi/4$ satisfies $\tan\psi = \left.\frac{r}{dr/d\theta}\right|_{\theta=\pi/4} = -1$, and $\psi = 3\pi/4$. The two curves intersect at P_1 at angle $\frac{3\pi}{4} - \frac{\pi}{2} = \frac{\pi}{4}$.

The Figure shows that at the origin, P_2, the circle meets the lemniscate twice, at angles 0 and $\pi/2$.

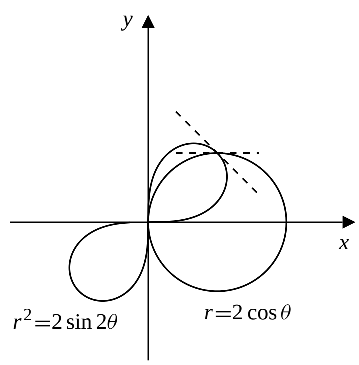

Fig. 6-18

11. $\dfrac{dy}{dx} = \dfrac{3y}{x-1} \Rightarrow \displaystyle\int \frac{dy}{y} = 3\frac{dx}{x-1}$

$$\Rightarrow \ln|y| = \ln|x-1|^3 + \ln|C|$$

$$\Rightarrow y = C(x-1)^3.$$

Since $y = 4$ when $x = 2$, we have $4 = C(2-1)^3 = C$, so the equation of the curve is $y = 4(x-1)^3$.

8.

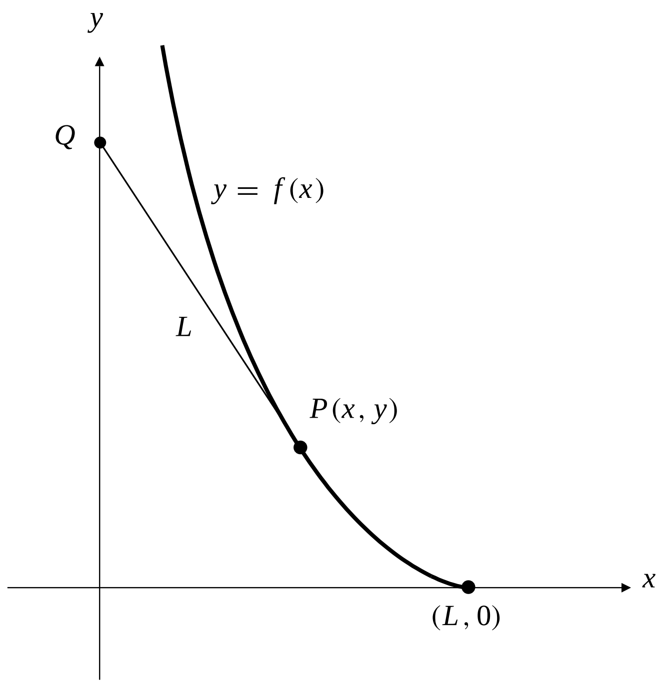

Fig. C-8

If $Q = (0, Y)$, then the slope of PQ is

$$\frac{y - Y}{x - 0} = f'(x) = \frac{dy}{dx}.$$

Since $|PQ| = L$, we have $(y - Y)^2 = L^2 - x^2$. Since the slope dy/dx is negative at P, $dy/dx = -\sqrt{L^2 - x^2}/x$. Thus

$$y = -\int \frac{\sqrt{L^2 - x^2}}{x}\, dx = L \ln\left(\frac{L + \sqrt{L^2 - x^2}}{x}\right) - \sqrt{L^2 - x^2} + C.$$

Since $y = 0$ when $x = L$, we have $C = 0$ and the equation of the tractrix is

$$y = L \ln\left(\frac{L + \sqrt{L^2 - x^2}}{x}\right) - \sqrt{L^2 - x^2}.$$

Note that the first term can be written in an alternate way:

$$y = L \ln\left(\frac{x}{L - \sqrt{L^2 - x^2}}\right) - \sqrt{L^2 - x^2}.$$

18. Since $f(x) = \dfrac{2}{\pi(1+x^2)} > 0$ on $[0, \infty)$ and

$$\frac{2}{\pi}\int_0^\infty \frac{dx}{1+x^2} = \lim_{R\to\infty} \frac{2}{\pi}\tan^{-1}(R) = \frac{2}{\pi}\left(\frac{\pi}{2}\right) = 1,$$

therefore $f(x)$ is a probability density function on $[0, \infty)$. The expectation of X is

$$\begin{aligned}\mu = E(X) &= \frac{2}{\pi}\int_0^\infty \frac{x\,dx}{1+x^2} \\ &= \lim_{R\to\infty} \frac{1}{\pi}\ln(1+R^2) = \infty.\end{aligned}$$

No matter what the cost per game, you should be willing to play (if you have an adequate bankroll). Your expected winnings per game in the long term is infinite.