16. Let a circular disk with radius a have centre at point $(a, 0)$. Then the disk is rotated about the y-axis which is one of its tangent lines. The volume is:

$$\begin{aligned} V &= 2 \times 2\pi \int_0^{2a} x\sqrt{a^2 - (x-a)^2}\,dx \qquad \text{Let } u = x - a \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad du = dx \\ &= 4\pi \int_{-a}^{a} (u+a)\sqrt{a^2 - u^2}\,du \\ &= 4\pi \int_{-a}^{a} u\sqrt{a^2 - u^2}\,du + 4\pi a \int_{-a}^{a} \sqrt{a^2 - u^2}\,du \\ &= 0 + 4\pi a\left(\frac{1}{2}\pi a^2\right) = 2\pi^2 a^3 \text{ cu. units.} \end{aligned}$$

(Note that the first integral is zero because the integrand is odd and the interval is symmetric about zero; the second integral is the area of a semicircle.)

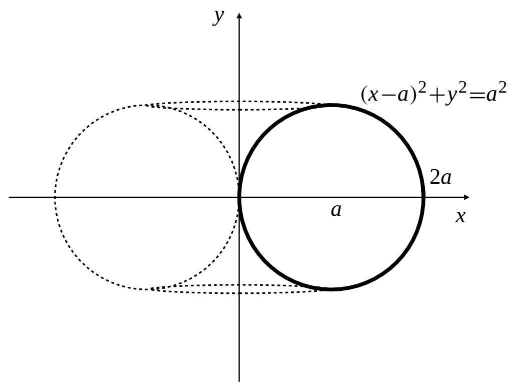

Fig. 1-16

29. The region is symmetric about $x = y$ so has the same volume of revolution about the two coordinate axes. The volume of revolution about the y-axis is

$$\begin{aligned}
V &= 2\pi \int_0^8 x(4 - x^{2/3})^{3/2}\,dx \qquad \text{Let } x = 8\sin^3 u \\
&\qquad\qquad\qquad\qquad\qquad\qquad dx = 24\sin^2 u \cos u\,du \\
&= 3072\pi \int_0^{\pi/2} \sin^5 u \cos^4 u\,du \\
&= 3072\pi \int_0^{\pi/2} (1 - \cos^2 u)^2 \cos^4 u \sin u\,du \qquad \text{Let } v = \cos u \\
&\qquad\qquad\qquad\qquad\qquad\qquad dv = -\sin u\,du \\
&= 3072\pi \int_0^1 (1 - v^2)^2 v^4\,dv \\
&= 3072\pi \int_0^1 (v^4 - 2v^6 + v^8)\,dv \\
&= 3072\pi \left(\frac{1}{5} - \frac{2}{7} + \frac{1}{9}\right) = \frac{8192\pi}{105} \text{ cu. units.}
\end{aligned}$$

9. The volume between height 0 and height z is z^3. Thus

$$z^3 = \int_0^z A(t)\,dt,$$

where $A(t)$ is the cross-sectional area at height t. Differentiating the above equation with respect to z, we get $3z^2 = A(z)$. The cross-sectional area at height z is $3z^2$ sq. units.

11. $V = 2\int_0^r \left(2\sqrt{r^2 - y^2}\right)^2 dy$

$$= 8\int_0^r (r^2 - y^2)\, dy = 8\left(r^2 y - \frac{y^3}{3}\right)\Bigg|_0^r = \frac{16r^3}{3} \text{ cu. units.}$$

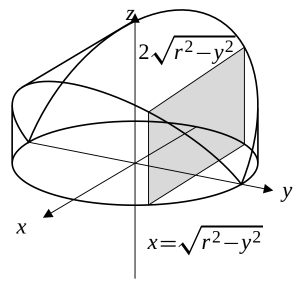

Fig. 2-11

12. The area of an equilateral triangle of base $2y$ is $\frac{1}{2}(2y)(\sqrt{3}y) = \sqrt{3}y^2$. Hence, the solid has volume

$$\begin{aligned} V &= 2\int_0^r \sqrt{3}(r^2 - x^2)\,dx \\ &= 2\sqrt{3}\Big(r^2x - \frac{1}{3}x^3\Big)\Big|_0^r \\ &= \frac{4}{\sqrt{3}}r^3 \text{ cu. units.} \end{aligned}$$

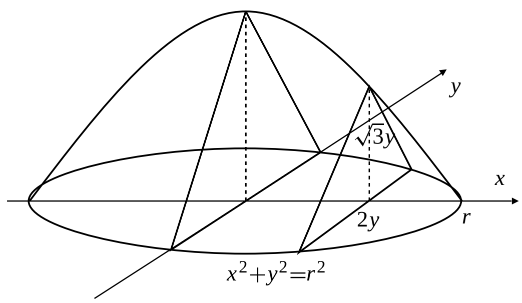

Fig. 2-12

13. $y = x^2,\ 0 \le x \le 2,\ y' = 2x.$

$$\begin{aligned} \text{length} &= \int_0^2 \sqrt{1+4x^2}\,dx \qquad \text{Let } 2x = \tan\theta \\ & \qquad\qquad\qquad\qquad\quad 2\,dx = \sec^2\theta\,d\theta \\ &= \frac{1}{2}\int_{x=0}^{x=2} \sec^3\theta \\ &= \frac{1}{4}\big(\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|\big)\Big|_{x=0}^{x=2} \\ &= \frac{1}{4}\big(2x\sqrt{1+4x^2} + \ln(2x+\sqrt{1+4x^2})\big)\Big|_0^2 \\ &= \frac{1}{4}\big(4\sqrt{17} + \ln(4+\sqrt{17})\big) \\ &= \sqrt{17} + \frac{1}{4}\ln(4+\sqrt{17}) \text{ units.} \end{aligned}$$

14. $y = \ln \dfrac{e^x - 1}{e^x + 1}, \qquad 2 \le x \le 4$

$$y' = \frac{e^x + 1}{e^x - 1} \, \frac{(e^x + 1)e^x - (e^x - 1)e^x}{(e^x + 1)^2}$$

$$= \frac{2e^x}{e^{2x} - 1}.$$

The length of the curve is

$$\begin{aligned}
L &= \int_2^4 \sqrt{1 + \frac{4e^{2x}}{(e^{2x} - 1)^2}} \, dx \\
&= \int_2^4 \frac{e^{2x} + 1}{e^{2x} - 1} \, dx \\
&= \int_2^4 \frac{e^x + e^{-x}}{e^x - e^{-x}} \, dx = \ln \left| e^x - e^{-x} \right| \Big|_2^4 \\
&= \ln \left(e^4 - \frac{1}{e^4} \right) - \ln \left(e^2 - \frac{1}{e^2} \right) \\
&= \ln \left(\frac{e^8 - 1}{e^4} \, \frac{e^2}{e^4 - 1} \right) = \ln \frac{e^4 + 1}{e^2} \text{ units.}
\end{aligned}$$

36.

$$
\begin{aligned}
S &= 2\pi \int_0^1 |x|\sqrt{1+\frac{1}{x^2}}\,dx \\
&= 2\pi \int_0^1 \sqrt{x^2+1}\,dx \qquad \text{Let } x = \tan\theta \\
&\qquad\qquad\qquad\qquad\qquad\ \ dx = \sec^2\theta\,d\theta \\
&= 2\pi \int_0^{\pi/4} \sec^3\theta\,d\theta \\
&= \pi\left(\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|\right)\Big|_0^{\pi/4} \\
&= \pi[\sqrt{2}+\ln(\sqrt{2}+1)] \text{ sq. units.}
\end{aligned}
$$

5. The mass of the plate is

$$m = 2\int_0^4 ky\sqrt{4-y}\,dy \qquad \text{Let } u = 4-y$$
$$\qquad\qquad\qquad\qquad du = -dy$$
$$= 2k\int_0^4 (4-u)u^{1/2}\,du$$
$$= 2k\left(\frac{8}{3}u^{3/2} - \frac{2}{5}u^{5/2}\right)\bigg|_0^4 = \frac{256k}{15}.$$

By symmetry, $M_{x=0} = 0$, so $\overline{x} = 0$.

$$M_{y=0} = 2\int_0^4 ky^2\sqrt{4-y}\,dy \qquad \text{Let } u = 4-y$$
$$\qquad\qquad\qquad\qquad du = -dy$$
$$= 2k\int_0^4 (16u^{1/2} - 8u^{3/2} + u^{5/2})\,du$$
$$= 2k\left(\frac{32}{3}u^{3/2} - \frac{16}{5}u^{5/2} + \frac{2}{7}u^{7/2}\right)\bigg|_0^4 = \frac{4096k}{105}.$$

Thus $\overline{y} = \frac{4096k}{105}\cdot\frac{15}{256k} = \frac{16}{7}$. The centre of mass of the plate is $(0, 16/7)$.

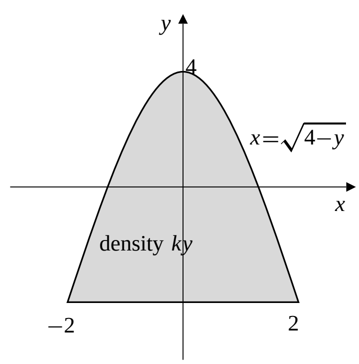

Fig. 4-5

11. Choose axes through the centre of the ball as shown in the following figure. The mass of the ball is

$$\begin{aligned} m &= \int_{-R}^{R} (y+2R)\pi(R^2-y^2)\,dy \\ &= 4\pi R\left(R^2y-\frac{y^3}{3}\right)\Bigg|_0^R = \frac{8}{3}\pi R^4 \text{ kg.} \end{aligned}$$

By symmetry, the centre of mass lies along the y-axis; we need only calculate $\overline{y}$.

$$\begin{aligned} M_{y=0} &= \int_{-R}^{R} y(y+2R)\pi(R^2-y^2)\,dy \\ &= 2\pi\int_0^R y^2(R^2-y^2)\,dy \\ &= 2\pi\left(R^2\frac{y^3}{3}-\frac{y^5}{5}\right)\Bigg|_0^R = \frac{4}{15}\pi R^5. \end{aligned}$$

Thus $\overline{y} = \dfrac{4\pi R^5}{15}\cdot\dfrac{3}{8\pi R^4} = \dfrac{R}{10}$. The centre of mass is on the line through the centre of the ball perpendicular to the plane mentioned in the problem, at a distance $R/10$ from the centre of the ball on the side opposite to the plane.

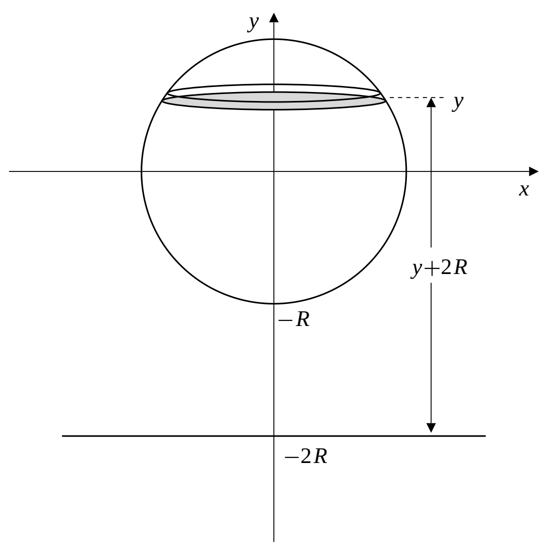

Fig. 4-11

12. A slice at height z has volume $dV = \pi y^2\,dz$ and density kz g/cm^3. Thus, the mass of the cone is

$$\begin{aligned} m &= \int_0^b kz\pi y^2\,dz \\ &= \pi k a^2 \int_0^b z\left(1 - \frac{z}{b}\right)^2 dz \\ &= \pi k a^2 \left(\frac{z^2}{2} - \frac{2z^3}{3b} + \frac{z^4}{4b^2}\right)\Bigg|_0^b \\ &= \frac{1}{12}\pi k a^2 b^2 \text{ g.} \end{aligned}$$

The moment about $z = 0$ is

$$M_{z=0} = \pi k a^2 \int_0^b z^2\left(1 - \frac{z}{b}\right)^2 dz = \frac{1}{30}\pi k a^2 b^3 \text{ g-cm.}$$

Thus, $\bar{z} = \dfrac{2b}{5}$. Hence, the centre of mass is on the axis of the cone at height $2b/5$ cm above the base.

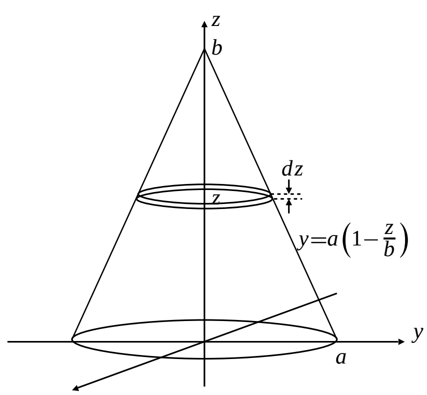

Fig. 4-12