15. $2f(x) + 1 = 3\int_x^1 f(t)\,dt$

$$2f'(x) = -3f(x) \quad \Longrightarrow \quad f(x) = Ce^{-3x/2}$$

$$2f(1) + 1 = 0$$

$$-\frac{1}{2} = f(1) = Ce^{-3/2} \quad \Longrightarrow \quad C = -\frac{1}{2}e^{3/2}$$

$$f(x) = -\frac{1}{2}e^{(3/2)(1-x)}.$$

15. The curves $y = \dfrac{4}{x^2}$ and $y = 5 - x^2$ intersect where $x^4 - 5x^2 + 4 = 0$, i.e., where $(x^2 - 4)(x^2 - 1) = 0$. Thus the intersections are at $x = \pm 1$ and $x = \pm 2$. We have

$$\begin{aligned}\text{Area of } R &= 2\int_1^2 \left(5 - x^2 - \frac{4}{x^2}\right) dx \\ &= 2\left(5x - \frac{x^3}{3} + \frac{4}{x}\right)\Bigg|_1^2 = \frac{4}{3} \text{ sq. units.}\end{aligned}$$

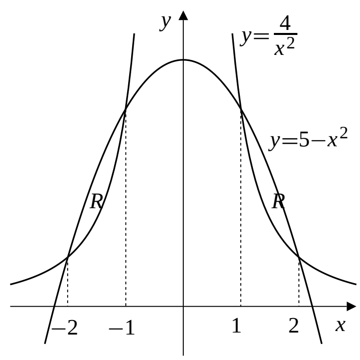

Fig. 7-15

28. Loop area $= 2\int_{-2}^{0} x^2\sqrt{2+x}\,dx$ Let $u^2 = 2+x$

$2u\,du = dx$

$$= 2\int_0^{\sqrt{2}} (u^2-2)^2 u(2u)\,du = 4\int_0^{\sqrt{2}} (u^6 - 4u^4 + 4u^2)\,du$$

$$= 4\left(\frac{1}{7}u^7 - \frac{4}{5}u^5 + \frac{4}{3}u^3\right)\Bigg|_0^{\sqrt{2}} = \frac{256\sqrt{2}}{105} \text{ sq. units.}$$

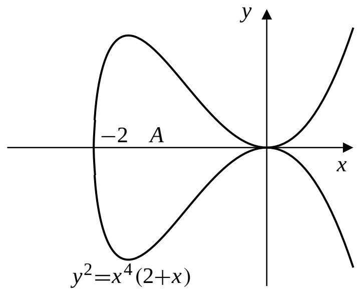

Fig. 7-28

25. $\int_0^4 \sqrt{9t^2 + t^4}\, dt$

$= \int_0^4 t\sqrt{9 + t^2}\, dt \quad$ Let $u = 9 + t^2$

$du = 2t\, dt$

$= \frac{1}{2}\int_9^{25} \sqrt{u}\, du = \frac{1}{3}u^{3/2}\Big|_9^{25} = \frac{98}{3}$

30. $$\int \cos^2 \frac{t}{5} \sin^2 \frac{t}{5}\, dt = \frac{1}{4} \int \sin^2 \frac{2t}{5}\, dt$$
$$= \frac{1}{8} \int \left(1 - \cos \frac{4t}{5}\right) dt$$
$$= \frac{1}{8} \left(t - \frac{5}{4} \sin \frac{4t}{5}\right) + C$$

28. $$\int \cos^4 x\,dx = \int \frac{[1+\cos(2x)]^2}{4}\,dx$$
$$= \frac{1}{4}\int [1 + 2\cos(2x) + \cos^2(2x)]\,dx$$
$$= \frac{x}{4} + \frac{\sin(2x)}{4} + \frac{1}{8}\int 1 + \cos(4x)\,dx$$
$$= \frac{x}{4} + \frac{\sin(2x)}{4} + \frac{x}{8} + \frac{\sin(4x)}{32} + C$$
$$= \frac{3x}{8} + \frac{\sin(2x)}{4} + \frac{\sin(4x)}{32} + C.$$

17. $\displaystyle\int \frac{dx}{e^x + 1} = \int \frac{e^{-x}\,dx}{1 + e^{-x}}$ Let $u = 1 + e^{-x}$

$du = -e^{-x}\,dx$

$$= -\int \frac{du}{u} = -\ln|u| + C = -\ln(1 + e^{-x}) + C.$$

47. Area $R = \int_0^2 \frac{x\,dx}{x^4 + 16}$ Let $u = x^2$
$du = 2x\,dx$

$$= \frac{1}{2}\int_0^4 \frac{du}{u^2 + 16} = \frac{1}{8}\tan^{-1}\frac{u}{4}\bigg|_0^4 = \frac{\pi}{32}\text{sq. units.}$$

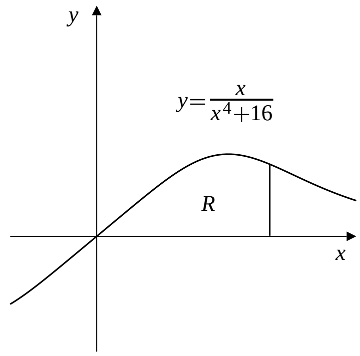

Fig. 6-47

45. $\int_0^{\pi/2} \sqrt{1+\cos x}\, dx = \int_0^{\pi/2} \sqrt{2\cos^2 \frac{x}{2}}\, dx$

$$= \sqrt{2}\int_0^{\pi/2} \cos\frac{x}{2}\, dx = 2\sqrt{2}\sin\frac{x}{2}\Big|_0^{\pi/2} = 2.$$

$$\int_0^{\pi/2} \sqrt{1-\sin x}\, dx$$

$$= \int_0^{\pi/2} \sqrt{1-\cos\left(\tfrac{\pi}{2}-x\right)}\, dx \qquad \text{Let } u = \tfrac{\pi}{2} - x$$

$$du = -dx$$

$$= -\int_{\pi/2}^{0} \sqrt{1-\cos u}\, du$$

$$= \int_0^{\pi/2} \sqrt{2\sin^2\frac{u}{2}}\, du = \sqrt{2}\left(-2\cos\frac{u}{2}\right)\Big|_0^{\pi/2}$$

$$= -2 + 2\sqrt{2} = 2(\sqrt{2}-1).$$

51. $F(x) = \int_0^{2x-x^2} \cos\left(\frac{1}{1+t^2}\right) dt.$

Note that $0 < \frac{1}{1+t^2} \le 1$ for all t, and hence

$$0 < \cos(1) \le \cos\left(\frac{1}{1+t^2}\right) \le 1.$$

The integrand is continuous for all t, so $F(x)$ is defined and differentiable for all x. Since $\lim_{x\to\pm\infty}(2x - x^2) = -\infty$, therefore $\lim_{x\to\pm\infty} F(x) = -\infty$. Now

$$F'(x) = (2-2x)\cos\left(\frac{1}{1+(2x-x^2)^2}\right) = 0$$

only at $x = 1$. Therefore F must have a maximum value at $x = 1$, and no minimum value.

32. $f(x) = 4x - x^2 \geq 0$ if $0 \leq x \leq 4$, and $f(x) < 0$ otherwise. If $a < b$, then $\int_a^b f(x)\,dx$ will be maximum if $[a, b] = [0, 4]$; extending the interval to the left of 0 or to the right of 4 will introduce negative contributions to the integral. The maximum value is

$$\int_0^4 (4x - x^2)\,dx = \left(2x^2 - \frac{x^3}{3}\right)\Big|_0^4 = \frac{32}{3}.$$

51. If m and n are integers, and $m \neq n$, then

$$\begin{aligned}
&\int_{-\pi}^{\pi} \left\{ \begin{matrix} \cos mx \cos nx \\ \sin mx \sin nx \end{matrix} \right\} dx \\
&= \frac{1}{2} \int_{-\pi}^{\pi} \big(\cos(m-n)x \pm \cos(m+n)x \big)\, dx \\
&= \frac{1}{2} \left(\frac{\sin(m-n)x}{m-n} \pm \frac{\sin(m+n)x}{m+n} \right) \Bigg|_{-\pi}^{\pi} \\
&= 0 \pm 0 = 0. \\
&\int_{-\pi}^{\pi} \sin mx \cos nx\, dx \\
&= \frac{1}{2} \int_{-\pi}^{\pi} \big(\sin(m+n)x + \sin(m-n)x \big)\, dx \\
&= -\frac{1}{2} \left(\frac{\cos(m+n)x}{m+n} + \frac{\cos(m-n)x}{m-n} \right) \Bigg|_{-\pi}^{\pi} \\
&= 0 \text{ (by periodicity).}
\end{aligned}$$

If $m = n \neq 0$ then

$$\begin{aligned}
&\int_{-\pi}^{\pi} \sin mx \cos mx\, dx \\
&= \frac{1}{2} \int_{-\pi}^{\pi} \sin 2mx\, dx \\
&= -\frac{1}{4m} \cos 2mx \Bigg|_{-\pi}^{\pi} = 0 \text{ (by periodicity).}
\end{aligned}$$

52. If $1 \leq m \leq k$, we have

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\cos mx\,dx = \frac{a_0}{2}\int_{-\pi}^{\pi}\cos mx\,dx \\ + \sum_{n=1}^{k} a_n \int_{-\pi}^{\pi}\cos nx\,\cos mx\,dx \\ + \sum_{n=1}^{k} b_n \int_{-\pi}^{\pi}\sin nx\,\cos mx\,dx.\end{aligned}$$

By the previous exercise, all the integrals on the right side are zero except the one in the first sum having $n = m$. Thus the whole right side reduces to

$$\begin{aligned}a_m \int_{-\pi}^{\pi}\cos^2(mx)\,dx &= a_m \int_{-\pi}^{\pi}\frac{1+\cos(2mx)}{2}\,dx \\ &= \frac{a_m}{2}(2\pi + 0) = \pi a_m.\end{aligned}$$

Thus

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx\,dx.$$

A similar argument shows that

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx\,dx.$$

For $m = 0$ we have

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\cos mx\,dx &= \int_{-\pi}^{\pi} f(x)\,dx \\ &= \frac{a_0}{2}\int_{-\pi}^{\pi} dx \\ &\quad + \sum_{n=1}^{k}\left(a_n\cos(nx) + b_n\sin(nx)\right)\,dx \\ &= \frac{a_0}{2}(2\pi) + 0 + 0 = a_0\pi,\end{aligned}$$

so the formula for a_m holds for $m = 0$ also.

5. We want to prove that for each positive integer k,

$$\sum_{j=1}^{n} j^k = \frac{n^{k+1}}{k+1} + \frac{n^k}{2} + P_{k-1}(n),$$

where P_{k-1} is a polynomial of degree at most $k-1$.
First check the case $k = 1$:

$$\sum_{j=1}^{n} j = \frac{n(n+1)}{2} = \frac{n^{1+1}}{1+1} + \frac{n}{2} + P_0(n),$$

where $P_0(n) = 0$ certainly has degree ≤ 0. Now assume that the formula above holds for $k = 1, 2, 3, \ldots, m$. We will show that it also holds for $k = m+1$. To this end, sum the the formula

$$(j+1)^{m+2} - j^{m+2} = (m+2)j^{m+1} + \frac{(m+2)(m+1)}{2} j^m + \cdots + 1$$

(obtained by the Binomial Theorem) for $j = 1, 2, \ldots, n$. The left side telescopes, and we get

$$\begin{aligned}(n+1)^{m+2} - 1^{m+2} &= (m+2)\sum_{j=1}^{n} j^{m+1} \\ &\quad + \frac{(m+2)(m+1)}{2}\sum_{j=1}^{n} j^m + \cdots + \sum_{j=1}^{n} 1.\end{aligned}$$

Expanding the binomial power on the left and using the induction hypothesis on the other terms we get

$$\begin{aligned}n^{m+2} + (m+2)n^{m+1} + \cdots &= (m+2)\sum_{j=1}^{n} j^{m+1} \\ &\quad + \frac{(m+2)(m+1)}{2}\,\frac{n^{m+1}}{m+1} + \cdots,\end{aligned}$$

where the $\cdots$ represent terms of degree m or lower in the variable n. Solving for the remaining sum, we get

$$\begin{aligned}&\sum_{j=1}^{n} j^{m+1} \\ &= \frac{1}{m+2}\left(n^{m+2} + (m+2)n^{m+1} + \cdots - \frac{m+2}{2}n^{m+1} - \cdots\right) \\ &= \frac{n^{m+2}}{m+2} + \frac{n^{m+1}}{2} + \cdots\end{aligned}$$

so that the formula is also correct for $k = m+1$. Hence it is true for all positive integers k by induction.