Math 440/508 Quiz 8 Solution

Name: **SID #:**

1. Explain whether there exists an analytic branch of the logarithm on the domain
$$\Omega = \mathbb{C} \setminus \{z = x + ix^2 : x \geq 0\}.$$
If yes, give an explicit formula for the logarithm.

(10 points)

Solution. The domain Ω is simply connected, so we know that there is an analytic branch of the logarithm. The issue is to define the function "$z \mapsto \arg(z)$" so that it is continuous on this domain.

The circle $|z| = r$ intersects the curve $\{x + ix^2 : x \geq 0\}$ at a unique point $z_0(r) = x_0(r) + i(x_0(r))^2$, where
$$x_0^2 + x_0^4 = r^2, \qquad \text{i.e.} \qquad x_0(r) = \frac{1}{\sqrt{2}}\sqrt{\sqrt{1+4r^2} - 1}.$$
Let $\theta_0(r)$ denote the unique value of $\arctan(x_0(r))$ that lies in $[0, \frac{\pi}{2})$.

Given $z = x + iy \in \Omega$ with $|z| = r$, we define
$$\arg_\Omega(z) = \theta,$$
where θ is the unique value of $\arctan(y/x)$ lying in $(\theta_0(r), \theta_0(r) + 2\pi)$. We note that this defines a continuous function on Ω. An analytic branch of the complex logarithm on Ω is then given by
$$\log_\Omega(z) = \log|z| + i \arg_\Omega(z).$$
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