Math 440/508 Quiz 6 Solution

Name: **SID #:**

1. If f is holomorphic on $0 < |z| < 2$ and satisfies $f(\frac{1}{n}) = n^2$ and $f(-\frac{1}{n}) = n^3$ for all positive integers n, what kind of singularity does f have at 0? Give complete justification for your answer.

(10 points)

Proof. The singularity is essential. Since f is unbounded near 0, it cannot have a removable singularity at the origin by Riemann's theorem. Therefore it only remains to eliminate the possibility of a pole.

Aiming for a contradiction, let us assume that f has a pole of order $m \geq 1$ at the origin. Thus, we may write

$$f(z) = \frac{g(z)}{z^m} \quad \text{where } g \text{ is holomorphic near the origin, with } g(0) \neq 0.$$

Set $z = 1/n$, which gives $n^2 = g(1/n)n^m$. Letting $n \to \infty$ and in view of the above requirement on g, we see that $m = 2$. On the other hand, if we set $z = -1/n$, the same procedure yields $-n^3 = g(-1/n)n^m$, which says that m must be 3. Since the order of a pole m is uniquely defined, this is a contradiction. □