Math 440/508 Quiz 5 Solution

Name: **SID #:**

1. Express the integral

$$\int_0^\pi \frac{d\theta}{a+\sin^2\theta} \quad \text{in the form} \quad \oint_{|z|=1} \frac{f(z)}{z-z_0}\,dz,$$

and then use the Cauchy integral formula to evaluate it. Here $a > 1$ is a fixed constant.

(10 points)

Solution. We write

$$\int_0^\pi \frac{d\theta}{a+\sin^2\theta} = \int_0^\pi \left[a+\frac{1}{2}(1-\cos(2\theta)\right]^{-1} d\theta = \frac{1}{2}\int_0^{2\pi}\left[a+\frac{1}{2}(1-\cos\varphi\right]^{-1} d\varphi$$
$$= \int_0^{2\pi} \left[(2a+1)-\cos\varphi\right]^{-1} d\varphi,$$

where the second step uses the change of variable $\varphi = 2\theta$. Set $z = e^{i\varphi}$. Then, $d\varphi = \frac{1}{i}\frac{dz}{z}$, and

$$\cos\varphi = \frac{e^{i\varphi}+e^{-\varphi}}{2} = \frac{z+1/z}{2}.$$

Substituting these into the integral above, we obtain

$$\int_0^\pi \frac{d\theta}{a+\sin^2\theta} = \frac{1}{i}\oint_{|z|=1} \frac{2dz}{2(2a+1)z-z^2-1} = -\frac{2}{i}\oint_{|z|=1}\frac{\frac{1}{z-z_1}dz}{z-z_0},$$

where

$$z_0 = 2a+1-2\sqrt{a^2+a}, \text{ and}$$
$$z_1 = 2a+1+2\sqrt{a^2+a}$$

are the two roots of the polynomial $z^2-2(2a+1)z+1$. Note that z_1 lies outside the unit disk and z_0 lies in its interior. Therefore by the Cauchy integral formula with $f(z) = 1/(z-z_1)$, we obtain

$$\int_0^\pi \frac{d\theta}{a+\sin^2\theta} = -\frac{2}{i}2\pi i\frac{1}{z_0-z_1} = \frac{\pi}{\sqrt{a^2+a}}.$$

□