Math 440/508 Quiz 4 Solution

1. Identify all entire functions f with the following property: $f(\mathbb{C}) \cap \mathbb{D} = \emptyset$ for some disk $\mathbb{D}$. (10 points)

Solution. Let $\mathbb{D} = \{z : |z - w_0| < r\}$. If f is an entire function such that $f(\mathbb{C}) \subseteq \mathbb{D}^c$, then

$$|f(z) - w_0| \geq r \quad \text{for all } z \in \mathbb{C}.$$

In particular, this means that $f(z) - w_0$ never vanishes, and therefore $g(z) = 1/(f(z) - w_0)$ is an entire function. Moreover, the inequality above gives that $|g(z)| \leq 1/r$, establishing that g in a bounded, entire function. By Liouville's theorem, g has to be a constant, hence so is f. $\square$