Math 440/508 Quiz 11 Solution

Name: **SID #:**

1. Suppose that f_n is a sequence of holomorphic functions on a complex domain G. Assume
   (a) $f_n \to f$ uniformly on all compact subsets of G, and
   (b) $f \not\equiv 0$.
   Fix any disc $\mathbb{D}$ such that $\overline{\mathbb{D}} \subseteq G$ and f is nonzero on $\partial\mathbb{D}$. Show that there exists $N \geq 1$ such that for every $n \geq N$, f_n and f share the same number of zeros in $\mathbb{D}$, counting multiplicity.

   (10 points)

Solution. One can use either Rouché's theorem or the argument principle for this problem. Here is the solution using Rouché's theorem.

Since f is nonzero on $\partial\mathbb{D}$, there exists $\delta > 0$ such that $|f(z)| \geq \delta$ for $z \in \partial\mathbb{D}$. Since $f_n \to f$ uniformly on $\partial\mathbb{D}$, there exists $N \geq 1$ such that

$$\sup_{z \in \partial\mathbb{D}} |f_n(z) - f(z)| < \delta \quad \text{for all } n \geq N.$$

Therefore,

$$|f_n(z) - f(z)| < \delta \leq |f(z)| \leq |f(z)| + |f_n(z)| \text{ on } \partial\mathbb{D},$$

which is the hypothesis of Rouché's theorem. The conclusion of the theorem is precisely the conclusion required by the problem. □