Math 440/508 Quiz 10 Solution

Name: **SID #:**

1. Determine whether the following statement is true or false, with complete justification. *Every automorphism of the extended complex plane is a Möbius transformation.*

   (10 points)

   Solution. The statement is true. If $\varphi \in \mathrm{Aut}(\mathbb{C}_\infty)$, then φ is holomorphic (hence continuous) at every point in $\mathbb{C}_\infty$. In particular, φ cannot have an essential singularity at any point, since an essential singularity of φ would be a point of discontinuity in $\mathbb{C}_\infty$. Hence φ must be meromorphic in $\mathbb{C}_\infty$. We proved in class that a meromorphic function on the extended complex plane must be a rational function, i.e.,

   $$\varphi(z) = \frac{P(z)}{Q(z)}, \quad \text{where } P \text{ and } Q \text{ are polynomials.}$$

   For φ to be a bijection on $\mathbb{C}_\infty$, we must have P and Q to be linear, otherwise the points 0 and ∞ respectively would have multiple pre-images under φ. Therefore $\varphi(z) = (az+b)/(cz+d)$ for $a, b, c, d \in \mathbb{C}$. Since φ is non-constant, we must have $ad - bc \neq 0$. In other words, φ is a Möbius transformation. $\square$