Math 440/508 Quiz 1 Solution, Fall 2017

1. Does there exist a function $f : \mathbb{C} \to \mathbb{C}$ that is holomorphic at every point on the unit circle $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ and not holomorphic anywhere else in the complex plane? If yes, provide such a function with complete justification. If not, explain why not.

(10 points)

Solution. **Yes, such a function exists.**

Consider the function $f(z) = (|z| - 1)^2$. Then f is infinitely real-differentiable for all $z \neq 0$. For such z,

$$\frac{\partial f}{\partial \bar{z}} = 2(|z| - 1)\frac{z}{\bar{z}},$$

which is nonzero unless $|z| = 1$. We have proved in class that a smooth function f is holomorphic if and only if it satisfies the Cauchy-Riemann equations, namely $\partial f/\partial \bar{z} = 0$. Therefore we conclude that f is holomorphic at $z_0 \in \mathbb{C} \setminus \{0\}$ if and only if z_0 satisfies $|z_0| = 1$. Further if $z_0 = 0$, a direct computation shows that

$$\lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{|h|^2 - 2|h|}{h} = -2 \lim_{h \to 0} \frac{|h|}{h}$$

does not exist, as can be seen by choosing sequences h approaching 0 along the real and imaginary axis respectively. This proves that f is not holomorphic at zero either, completing the proof. □