Chapter 3, Exercise 14

Prove that all entire functions that are also injective take the form $f(z) = az + b$ with $a, b \in \mathbb{C}$ and $a \neq 0$.

Solution

Assume f is an entire injective function. Then f is nonconstant, so $g(z) := f(1/z)$ has either a pole or an essential singularity at $z = 0$. We will show first that the singularity at 0 cannot be an essential singularity. If it were an essential singularity, then the Cazorati-Weierstrass theorem would imply that the set $g(B(0,1) \setminus \{0\})$ is dense in $\mathbb{C}$. However, $g(B(2, \frac{1}{2}))$ is an open set by the open mapping theorem. Therefore these two sets intersect, which shows that $g(z)$ and hence $f(z)$ is not injective.

Therefore, the singularity at $z = 0$ must be a pole, implying that $f(z)$ is a polynomial. Suppose $f(z)$ is a polynomial of degree m. Then f has m roots, counting multiplicity. Evidently, if f has two distinct roots, then f is not injective. Thus $f(z) = c(z - z_0)^m$ for some complex numbers c and z_0. However, for $m \geq 2$ such functions are also non-injective: $f(z_0 + 1) = c = f(z_0 + e^{2\pi i/m})$. Thus $m = 1$ and $f(z)$ is a linear polynomial (evidently $c \neq 0$ since f is nonconstant).

Chapter 3, Exercise 22

Show that there is no holomorphic function f in the unit disc D that extends continuously to ∂D such that $f(z) = 1/z$ for each $z \in \partial D$.

Solution

We will abuse notation a bit and let f be the continuous extension of this function to $\overline{D}$. Notice that $f(z)$ is uniformly continuous on $\overline{D}$ since $\overline{D}$ is compact. By uniform continuity we have

$$\lim_{r \to 1} \int_{|z|=r} f(z)\, dz = \int_{|z|=1} f(z)\, dz.$$

We know that each integral $\int_{|z|=r} f(z)\, dz$ is zero because f is holomorphic inside D. However, we also know that

$$\int_{|z|=1} f(z)\, dz = 2\pi i$$

because $f(z) = \frac{1}{z}$ on the circle where $|z| = 1$. This is a contradiction, so no such function f can exist.

Chapter 3, Problem 3

Suppose f is holomorphic in a region containing the annulus $\{z : r_1 \le |z - z_0| \le r_2\}$ where $0 < r_1 < r_2$.

Show that

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n,$$

where the series converges absolutely in the interior of the annulus.

Proof

Fix z, and let C_1 and C_2 be the inner and outer circles, respectively. Draw two close, parallel line segments S_1 and S_2, separated by a distance δ, connecting C_1 and C_2 so that the point z is not contained in the small region bounded between the line segments S_1 and S_2. Without loss of generality suppose that the minor arc between S_1 and S_2 runs clockwise from S_1 to S_2 along both C_1 and C_2.

Consider the contour Γ starting at the intersection point of C_2 and S_1, moving counterclockwise along the major arc of C_2 until running into the intersection of S_2 and C_2, then travelling along S_2 until the intersection of S_2 and C_1, then travelling clockwise around C_1 until reaching the intersection of S_1 and C_1, then travelling along S_1 until reaching the intersection of S_1 and C_2. This is a closed contour that encloses z but not z_0. By using independence of path together with the Cauchy integral formula, we get

$$\frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta = f(z).$$

By taking the distance δ to go to zero, we see by uniform continuity that

$$\frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta = \frac{1}{2\pi i} \int_{C_2} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{2\pi i} \int_{C_1} \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

We will use this identity to write a Laurent series expansion for $f(z)$ centered at z_0. First we will consider the C_2 integral,

$$\frac{1}{2\pi i} \int_{C_2} \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

In order to expand this integral as a series, we rewrite

$$\begin{aligned}
\frac{f(\zeta)}{\zeta - z} &= \frac{f(\zeta)}{\zeta - z_0 + z_0 - z} \\
&= \frac{f(\zeta)}{\zeta - z_0} \left(\frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} \right) \\
&= \frac{f(\zeta)}{\zeta - z_0} \sum_{j=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0} \right)^j .
\end{aligned}$$

When we integrate this over C_2, we get

$$\frac{1}{2\pi i}\int_{C_2}\frac{f(\zeta)}{\zeta - z_0}\sum_{j=0}^{\infty}\left(\frac{z-z_0}{\zeta-z_0}\right)^j d\zeta.$$

We would like to interchange the integral and the sum. To do this, we appeal to the dominated convergence theorem: because $|\zeta - z_0|$ is greater than $|z - z_0|$ for $\zeta \in C_2$, It follows that the series inside the integral is absolutely convergent (and bounded above by the value $\left|\frac{f(\zeta)}{\zeta - z_0}\right| \cdot \frac{1}{1-\left|\frac{z-z_0}{\zeta-z_0}\right|}$), which in turn is bounded above by a constant value not depending on ζ for $\zeta \in C_2$. Therefore, the dominated convergence theorem applies and we can interchange the order of the integral and the sum.

$$\sum_{j=0}^{\infty}(z-z_0)^j\frac{1}{2\pi i}\int_{C_2}\frac{f(\zeta)}{\zeta - z_0}\left(\frac{z-z_0}{\zeta-z_0}\right)^j d\zeta.$$

We thus take

$$a_j = \frac{1}{2\pi i}\int_{C_2}\frac{f(\zeta)}{\zeta - z_0}\left(\frac{z-z_0}{\zeta-z_0}\right)^j d\zeta$$

giving the positive-degree terms in the Laurent series expansion. The sum is absolutely convergent by an argument similar to the one allowing us to apply the dominated convergence theorem.

A similar strategy for C_1 gives the negative-degree terms in the Laurent series expansion:

$$\begin{aligned}\frac{f(\zeta)}{\zeta - z} &= \frac{f(\zeta)}{\zeta - z_0 + z_0 - z}\\ &= \frac{-f(\zeta)}{z - z_0}\frac{1}{1-\frac{\zeta - z_0}{z-z_0}}\\ &= \frac{-f(\zeta)}{z - z_0}\sum_{j=0}^{\infty}\left(\frac{\zeta-z_0}{z-z_0}\right)^j\end{aligned}$$

Leaving us with the integral

$$-\frac{1}{2\pi i}\int_{C_1}\frac{-f(\zeta)}{z - z_0}\sum_{j=0}^{\infty}\left(\frac{\zeta-z_0}{z-z_0}\right)^j.$$

By a similar argument to the one given before, the sum inside the integral is absolutely convergent and the integrand is uniformly bounded, so the dominated convergence theorem applies, and we can pull the sum out of the integral:

$$\sum_{j=0}^{\infty}\frac{f(\zeta)}{(z-z_0)^{j+1}}\frac{1}{2\pi i}\int_{C_1}(\zeta - z_0)^j$$

so, for $k \leq -1$, taking

$$a_k = \frac{1}{2\pi i} \int_{C_1} (\zeta - z_0)^{-(k+1)}$$

gives the desired negative-degree terms in the Laurent series expansion. The sum is absolutely convergent by an estimate similar to the one used to apply the dominated convergence theorem.