Chapter 1, Exercise 22

Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ denote the set of positive integers. A subset $S \subset \mathbb{N}$ is said to be in arithmetic progression if

$$S = \{a, a + d, a + 2d, a + 3d, \ldots\}$$

where $a, d \in \mathbb{N}$. Here d is called the step of S.

Show that $\mathbb{N}$ cannot be partitioned into a finite number of subsets that are in arithmetic progression with distinct steps (except for the trivial case $a = d = 1$). [Hint: write $\sum_{n \in \mathbb{N}} z^n$ as a sum of terms of the type $\frac{z^a}{1-z^d}$.]

Solution

Suppose that we can partition $\mathbb{N}$ into disjoint arithmetic progressions $S_1, \ldots, S_n$, where S_j has smallest element a_j and step d_j. Then we have

$$\sum_{j=1}^{\infty} z^j = \frac{z}{1-z}$$

for all $|z| < 1$. We also have

$$\begin{aligned}
\sum_{j \in S_j} z^j &= \sum_{k=1}^{\infty} z^{a_j + kd_j} \\
&= z^{a_j} \sum_{k=0}^{\infty} z^{kd_j} \\
&= \frac{z^{a_j}}{1 - z^{d_j}}
\end{aligned}$$

Since the S_j form a partition of $\mathbb{N}$, we get, for $|z| < 1$, that

$$\frac{z}{1-z} = \sum_{j=1}^{n} \frac{z^{a_j}}{1 - z^{d_j}}.$$

Without loss of generality, assume d_1 is the largest of $\{d_1, \ldots, d_n\}$, Because the d_j are distinct, it follows that $d_j < d_1$ for $j \neq 1$. Consider the root of unity $\zeta = e^{2\pi i/d_1}$. For this root of unity we have $1 - z^{d_1} = 0$, but $1 - z^{d_j} \neq 0$ for any $j \neq d_j$. Thus the right hand side of the above equation is unbounded as $z \to \zeta$ from inside the unit disk. But $1 - \zeta \neq 0$ so the left hand side remains bounded as $z \to \zeta$ from inside the unit disk. Therefore, the left hand side is not equal to the right hand side and no such partition exists.

Chapter 1, Exercise 25

Part a

Evaluate the integrals

$$\int_\gamma z^n$$

for all integers n. Here γ is any circle centered at the origin with the positive (counterclockwise) orientation.

Solution

We can parameterize γ via the parameterization $z(t) = e^{2\pi i t}$, where t goes from 0 to 1. Under this parameterization, the integral becomes:

$$2\pi i \int_0^1 e^{2\pi i n t} \cdot e^{2\pi i t}\, dt.$$

Which is

$$2\pi i \int_0^1 e^{2\pi i (n+1) t}\, .dt$$

This integral can be written as

$$2\pi i \int_0^1 \cos(2\pi(n+1)t) + i\sin(2\pi(n+1)t)\, dt$$

Both trigonometric functions will run through an integer number of periods, and therefore the integral will vanish unless $n+1=0$, in which case the integrand is 1 and the integral evaluates to $2\pi i$. So for integers $n \neq -1$ the integral is zero; for $n=-1$ the integral is $2\pi i$.

Part b

Same question as before, but with γ any circle not enclosing the origin.

Solution

Let γ be any such circle. We can parameterize γ with the parameterization $\gamma(t) = z_0 + re^{2\pi i t}$ where $|z_0| > r$. In the region enclosed by γ, the function z^n has the explicit primitive $\frac{1}{n+1} z^{n+1}$ for $n \neq -1$. Therefore, the integral along γ is zero by Corollary 3.3.

If $n=-1$, then we have, after parametrizing,

$$\int_0^1 (z_0 + re^{2\pi i t})^{-1} e^{2\pi i t}.$$

This is

$$\int_0^1 z_0(1 + r/z_0 e^{2\pi i t})^{-1} e^{2\pi i t}.$$

Since $|r/z_0| < 1$, this has a power series expansion

$$\int_0^1 z_0 \sum_{j=0}^{\infty} (-r/z_0)^j e^{2\pi i(j+1)t}$$

and by absolute convergence of the sum, we can interchange the sum and integral. Because $j+1$ is a positive integer for any nonnegative integer j, each term in the sum integrates to 0 and the integral is 0.

Part c

Show that if $|a| < r < |b|$, then

$$\int_\gamma \frac{1}{(z-a)(z-b)}\, dz = \frac{2\pi i}{a-b},$$

where γ denotes the counterclockwise circle centered at the origin, of radius r, with the positive orientation.

Solution

For this problem, it is best to perform a partial fraction decomposition on $\frac{1}{(z-a)(z-b)}$. We rewrite

$$\frac{1}{(z-a)(z-b)} = \frac{1}{a-b}\left(\frac{1}{z-a} - \frac{1}{z-b}\right).$$

We then evaluate

$$\frac{1}{a-b}\int_\gamma \frac{1}{z-a}\, dz - \frac{1}{a-b}\int_\gamma \frac{1}{z-b}\, dz.$$

The second integral vanishes, as we computed in part b. For the first integral, we parameterize γ by $\gamma = re^{2\pi i\theta}$:

$$\frac{r2\pi i}{a-b}\int_0^1 (re^{2\pi it} - a)^{-1} e^{2\pi it}\, dt$$

Now, $r > |a|$, so we can pull out $re^{2\pi it}$ to get

$$\frac{r2\pi i}{a-b}\int_0^1 -\frac{1}{re^{2\pi it}}(1 - \frac{a}{r}e^{-2\pi it})^{-1} e^{2\pi it}\, dt$$

which gives

$$\frac{r2\pi i}{a-b}\int_0^1 \frac{1}{r}\sum_{j=0}^{\infty} a^j r^{-j} e^{-2\pi it(j+1)}\, dt$$

As before this integral vanishes unless $j = 0$, so we get

$$\frac{r2\pi i}{r(a-b)} = \frac{2\pi i}{a-b}.$$

Chapter 2, Problem 1

Here are some examples of analytic functions on the unit disc that cannot be extended analytically past the unit circle. The following definition is needed. Let f be a function defined in the unit disc $\mathbb{D}$, with boundary circle C. A point w on C is said to be *regular* for f if there is an open neighborhood U of w and an analytic function g on U, such that $f = g$ on $\mathbb{D} \cap U$. A function f defined on $\mathbb{D}$ cannot be continued analytically past the unit circle if no point of C is regular for f.

Part a

Let

$$f(z) = \sum_{n=0}^{\infty} z^{2^n} \text{ for } |z| < 1.$$

Notice that the radius of convergence of the above series is 1. Show that f cannot be continued analytically past the unit disc. [Hint: Suppose $\theta = 2\pi p/2^k$, where p and k are positive integers. Let $z = re^{i\theta}$; then $|f(re^{i\theta})| \to \infty$ as $r \to 1$.]

Solution

Let $\theta = \frac{2\pi p}{2^k}$, and let $z = re^{i\theta}$. Then we have

$$f(z) = \sum_{n=0}^{k-1} r^{2^n} e^{\frac{2\pi i * 2^n p}{2^k}} + \sum_{n=k}^{\infty} r^{2^n}.$$

Where we used the fact that the exponent was an integer multiple of $2\pi i$ for $n \geq k$. There are only k summands in the first sum, so the first sum is bounded by k in absolute value for $|r| < 1$. As $r \to 1$ from the left, the sum is therefore bounded from below by $\sum_n r^{2^n} - k$, which is unbounded as $r \to 1$. Thus $f(z)$ cannot be analytically extended to any ball containing a point of the form $e^{2\pi i p/2^k}$. Since points of this form are dense in the unit circle, $f(z)$ is not regular at any point of the unit circle.

Part b

Fix $0 < \alpha < \infty$. Show that the analytic function f defined by

$$f(z) = \sum_{n=0}^{\infty} 2^{-n\alpha} z^{2^n} \text{ for } |z| < 1$$

can be continuously extended to the unit circle, but cannot be analytically continued past the unit circle. [Hint: There is a nowhere differentiable function lurking in the background. See Chapter 4 in Book I.]

Solution

The function $f(z) = \sum_{n=0}^{\infty} 2^{-n\alpha} z^{2^n}$ converges absolutely and uniformly in the disc $|z| < 1$; by Abel's theorem, the function f can be extended continuously to the unit circle and is given by $f(z) = \sum_{n=0}^{\infty} 2^{-n\alpha} z^{2^n}$ on the unit circle $|z| \leq 1$. For $0 < \alpha < 1$, The function $f(z)$, restricted to the unit circle $z = e^{2\pi i\theta}$, is given by $\sum_{n=0}^{\infty} \cos(2^n 2\pi\theta) + i\sin(2^n 2\pi\theta)$. This is a nowhere differentiable Weierstrass function. Thus the function $f(z)$ is nowhere differentiable on the unit circle and therefore cannot be analytically extended past the circle.

For non-integer $\alpha > 1$, we can define the operator T by $Tf(z) = zf'(z)$. Letting $k < \alpha < k+1$, we take the function $T^k f$, which an be expanded for $z < 1$ as a power series:

$$Tf(z) = \sum_{n=0}^{\infty} 2^{-n(\alpha-k)} z^{2^n}.$$

By the previous argument, this cannot be analytically continued past the unit circle. Since $Tf(z)$ is analytic wherever $f(z)$ is analytic, it follows that $f(z)$ cannot be extended past the unit circle either.

For integers $\alpha = k$, and for $|z| < 1$, $T^k f$ is given by

$$\sum_{n=0}^{\infty} z^{2^n}$$

which cannot be analytically extended past the unit circle by part a.