Math 538, Lecture 17, 13/3/2024

Last time: The different

L/K finite extension of #fields/fields complete wrt discrete valuation.

Different is the inverse of the dual of $\mathcal{O}_L$ wrt trace form $\mathrm{Tr}^L_K(\cdot,\cdot)$

L/K unram $\Rightarrow$ $D_{L/K} = (1)$,

$\mathcal{O}_L = \mathcal{O}_K[\alpha]$ then $D_{L/K} = (f'(\alpha))$, f = min poly.

L, K #fields, $D_{L/K} = \prod_{v \in |K|_f} \cdot \prod_{w|v} D_{L_w/K_v}$.

("local-to-global")

Prop: Suppose L_w/K_v extension of complete fields, residue fields perfect. Let $\mathfrak{P}_w, \mathfrak{p}_v$ be the primes, e ramification index.

Then: (1) if extension is at most tamely ramified, $D_{L_w/K_v} = \mathfrak{P}_w^{e-1}$.

(2) if wildly ramified. $\mathfrak{P}_w^{e} \mid D_{L_w/K_v}$

(in either case $\mathfrak{P}_w^{e-1} \mid D_{L_w/K_v}$)

Pf: (1) Know this for unramified extensions

(2) By multiplicativity in towers, if M is unram closure of K_v in L_w, $D_{L_w/K_v} = D_{L_w/M}$

So may assume L_w/K_v is totally ramified

Then $L_w = K_v(\pi)$, π uniformiser satisfies Eisenstein poly $f(x) = x^e + \sum_{i=0}^{e-1} a_i x^i$, $a_i \in \mathfrak{p}_v$, $a_0 \notin \mathfrak{p}_v^2$

Also,

$\mathcal{O}_{L_w} = \mathcal{O}_{K_v}[\pi]$:

HW: Let $A \subset \mathcal{O}_{L_w}$ be a set of representatives for $\lambda_w = \mathcal{O}_{L_w}/\mathfrak{P}_w$. Then

$$\mathcal{O}_{L_w} = \left\{ \sum_{i=0}^{\infty} a_i \pi^i \;\middle|\; a_i \in A \right\}$$

Since L_w/K_v is totally ramified, $\lambda_w = k_v$ so can choose $A \subset \mathcal{O}_{K_v}$ $\Rightarrow$ $\mathcal{O}_K[\pi]$ is dense in $\mathcal{O}_L$

But $\mathcal{O}_{K_v}[\pi] \cong \mathcal{O}_{K_v}^e$ is cpt hence closed, so $\mathcal{O}_{K_v}[\pi] = \mathcal{O}_{L_w}$.

$$\Rightarrow D_{L_w/K_v} = (f'(\pi))$$

$$f'(\pi) = e\pi^{e-1} + \sum_{i=1}^{e-1} i a_i \pi^{i-1}.$$

now $a_i \in \mathfrak{p}_v = (\mathfrak{P}_w)^e$ so $\pi^e \mid a_i$ for each i.

If e is prime to π, get $\pi^{e-1} \parallel f'(\pi)$ ("tame ramification")

if not, $\pi^e \mid f'(\pi)$. ∎

⇒ Theorem: L/K fields. $\mathfrak{P} \triangleleft \mathcal{O}_L$ lying above $\mathfrak{p} \triangleleft \mathcal{O}_K$. Then $v_{\mathfrak{P}}(\mathcal{D}_{L/K}) \geq e(\mathfrak{P}/\mathfrak{p}) - 1$.
p rational prime below

(1) Equality if $p \nmid e(\mathfrak{P}/\mathfrak{p})$ (including $e=1$)
(2) strict inequality if $p \mid e$.

Cor: At most finitely many ramified primes

(if $\mathfrak{P}$ ramified, $\mathfrak{P} \mid \mathcal{D}_{L/K}$)

Remark: Since $\mathcal{D}_{L/K} \mid f'(\alpha)\mathcal{O}_L$ for any $\alpha \in \mathcal{O}_L$ s.t. $L = K(\alpha)$

$\mathfrak{P}$ ramified ⇒ $\mathfrak{P} \mid f'(\alpha)$.

The discriminant

Facts: $D_{L/K} = N^L_K \mathcal{D}_{L/K}$. then $\mathfrak{p} \triangleleft \mathcal{O}_K$ ramifies in L iff $\mathfrak{p} \mid D_{L/K}$.

① L/K be a finite separable extension of fields

Let $\Omega = \{\omega_i\}_{i \leq n}^{n} \subset L$ be a K-basis

Let $\{\sigma_j\}_{j=1}^n = \mathrm{Hom}_K(L, \bar{K})$

<u>Def</u>: $D_{L/K}(\mathcal{R}) = \det((\sigma_j w_i)_{ij})^2$

<u>Observe</u>: renumbering $\{w_i\}$, $\{\sigma_j\}$ amount to multiplying $(\sigma_j w_i)_{ij}$ by a permutation matrix

$\Rightarrow$ changes its det by ± 1.

Applying $\rho \in Gal(K^{sep}/K)$ to $D_{L/K}$ amounts to permuting σ_j: replace σ_j with $\rho \circ \sigma_j$

$\Rightarrow D_{L/K}(\mathcal{R}) \in K^{sep}$ is $Gal(K^{sep}/K)$

$\Rightarrow D_{L/K}(\mathcal{R}) \in K$

Let $a_{ij} = \sigma_j w_i$, $A = (a_{ij}) \in M_n(K^{sep})$

Let $\mathcal{R}' = \{w_k'\}_{k=1}^n \subset L$ be another basis.
Then have $S \in GL_n(K)$ st.

$$w_i = \sum_k s_{ik} w_k'$$

let $b_{kj} = \sigma_j w'_k$, $B = (b_{kj})_{kj}$, then

$$A = SB \Rightarrow \det(A)^2 = (\det S)^2 (\det B)^2$$

$$\Rightarrow D_{L|K}(\Omega) = (\det S)^2 D_{L/K}(\Omega')$$

<u>Claim:</u> $D_{L/K}(\Omega) \neq 0$

$\Rightarrow$ Can define $D_{L/K} \in K^\times/(K^\times)^2$ as class of any $D_{L/K}(\Omega)$

$\Rightarrow$ if Ω, Ω' generate same R-module for some $R \subset K$ then $\det(S) \in R^\times$, so given free R-submodule of L generated by basis get an invariant in $K^\times/(R^\times)^2$.

<u>Examples</u> L # field, $K = \mathbb{Q}$, $R = \mathbb{Z}$, module is $\mathcal{O}_L$. Get invariant "absolute discriminant" in $\mathbb{Z}_{\geq 1}$ (note: $(\mathbb{Z}^\times)^2 = 1$)

<u>HW:</u> For $\beta \in L^\times$, $D_{L/K}(\beta\Omega) = (N^L_K \beta)^2 D_{L/K}(\Omega)$

lemma: Let Ω, Ω' be two base, A, B associated matrices. Then

$$(AB^t)_{ik} = T^L_K(w_i w'_k)$$

Pf:

$$(AB^t)_{ik} = \sum_j a_{ij} b_{kj} = \sum_j (\sigma_j w_i) \cdot (\sigma_j w'_k)$$

$$= \sum_j \sigma_j (w_i w'_k) = T^L_K(w_i w'_k)$$

$\blacksquare$

Cor: AA^t is the Gram matrix of the trace form:

$$(AA^t)_{ik} = (w_i, w_k)_{Tr}.$$

$$\left(\sum_i x_i w_i, \sum_k y_k w_k\right) = \underline{x}^t AA^t \underline{y}$$

Cor: if Ω' is the basis dual to Ω wrt trace form, $AB^t = I_n$

$$\Rightarrow \det(A) \cdot \det(B) = 1$$

so $\det(A) \neq 0$. (and $\det(A)^2 = \det(AA^t)$
$= \det(\text{Gram matrix})$)

Lemma: let $L = K(\alpha)$, $\{\alpha_j\}_{j=1}^{n} \subset K^{sep}$ the Galois conjugates. Let $\Omega = \{\alpha^i\}_{i=0}^{n-1}$.

Then $D_{L/K}(\Omega) = \prod_{j<k} (\alpha_j - \alpha_k)^2 = \Delta(f)$

f = min poly of α $= \prod_{j=1}^{n} (x - \alpha_j)$

Pf: $a_{ij} = \sigma_j(\alpha^i) = (\sigma_j(\alpha))^i = \alpha_j^i$.

Claim is the Vandermonde determinant:

$\det(A) = \prod_{j<k} (\alpha_j - \alpha_k)$. ∎

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② Number fields

Assume L/K finite extension of # fields or fields complete wrt discrete valuation

Lemma: let $\mathfrak{a} \subset L$ be a fractional ideal. Then $\{D_{L/K}(\Omega) \mid \Omega \subset \mathfrak{a}\}$ generates a fractional ideal $D_{L/K}(\mathfrak{a}) \subset K$.

Def: Call $D_{L/K}(\mathfrak{a})$ the relative discriminant of $\mathfrak{a}$.

Pf: If $\mathfrak{a} \subset \mathcal{O}_L$ then $Tr_K^L(\omega_i\omega_j) \in \mathcal{O}_K$ for all i,j.

$\Rightarrow D_{L/K}(\mathfrak{a}) \in \mathcal{O}_K$.

$\Rightarrow$ if $\mathfrak{a} \trianglelefteq \mathcal{O}_L$ then $D_{L/K}(\mathfrak{a}) \trianglelefteq \mathcal{O}_K$

if a fractional ideal $\mathfrak{a} = \beta\mathfrak{b}$: $\mathfrak{b}$ ideal $\beta \in L^\times$.

$\Rightarrow D_{L/K}(\mathfrak{a}) = (N_K^L\beta)^2 \cdot D_{L/K}(\mathfrak{b})$ is a fractional ideal of K.

Observe: if $\mathfrak{a} \subset \mathfrak{b}$ $D_{L/K}(\mathfrak{a}) \subset D_{L/K}(\mathfrak{b})$

$\Leftrightarrow$

$\mathfrak{b} \mid \mathfrak{a} \Rightarrow D_{L/K}(\mathfrak{b}) \mid D_{L/K}(\mathfrak{a})$

Ex: If $\mathfrak{a} = \bigoplus_{i=1}^{n} \mathcal{O}_K \omega_i$ happens to be a free $\mathcal{O}_K$-mod

then $D_{L/K}(\mathfrak{a}) = \mathcal{O}_K \cdot D_{L/K}(\mathfrak{a})$

Def: The relative discriminant is $D_{L/K} = D_{L/K}(\mathcal{O}_L)$.

Cor: let $\alpha \in \mathcal{O}_L$ be s.t. $L = K(\alpha)$. Then

$D_{L/K} \mid D_{L/K}(\mathcal{O}_K[\alpha]) = (\Delta(f))$.

$f \in \mathcal{O}_K[x]$ = min poly of α.

<u>Prop</u>: L/K ext'n of # fields, $v \in |K|_f$
Then closure of $D_{L/K}$ in $\mathcal{O}_v$ is $\prod_{w|v} D_{L_w/K_v}$.

<u>Pf</u>: Recall $L \otimes_K K_v \cong \bigoplus_{w|v} L_w$

let's verify:
image of $\mathcal{O}_{K_v}$ on right is $\bigoplus_{w|v} \mathcal{O}_w$:

Let $S = \{w | v\} \subset |L|_f$, so $\mathcal{O}_L^S$ is dense in $\bigoplus_w L_w$.

Any $\alpha \in \mathcal{O}_L^S$ can be approximated by an element of $\mathcal{O}_L$ modulo a large power of primes in S.

Cor: Let $\alpha \in \mathcal{O}_L^S$ be very close to $(x_w) \in \bigoplus_w \mathcal{O}_w$.
Then α is S-integral and w-integral for all w
(so $\alpha \in \mathcal{O}_L$).

Now let $\Omega_w \subset \mathcal{O}_w$ be K_v-bases of L_w.
Then $\Omega = \bigcup_{w|v} \Omega_w$ is a K_v-basis of $K \otimes_K K_v$.

let $\Omega' \subset \mathcal{O}_L$ consist of elements close to Ω

Then α' is a basis and

$D_{L/K}(\alpha')$ is close to $D_{L/K}(\alpha) = \prod_w D_{L_w/K_v}(\alpha_w)$

$\Rightarrow \overline{D_{L/K}} \supset \prod_w D_{L_w/K_v}$

Discussion: Quad space assoc to trace form of K_v-algebra $K_v \otimes_K L$ is the ~~or~~ orthogonal sum of the spaces

$\{L_w\}_{w|v}$.