Math 538 Lecture 12, 16/2/2024

Last time: Ramification

K complete wrt non-arch non-trivial absolute value, $\mathcal{O}_K = \{x : |x| \leq 1\}$, $\mathfrak{p}_K = \{x : |x| < 1\}$, $\kappa = \mathcal{O}_K / \mathfrak{p}_K$.

L/K [finite]: $e(L/K) = [|L^\times| : |K^\times|]$
$f(L/K) = [\lambda : \kappa]$

Thm: $ef \leq n = [L:K]$; equality if $|K^\times| \subset \mathbb{R}^\times_{>0}$ is discrete.

Today: unramified, tamely ramified, wildly ramified extensions

Def: A finite extension L/K is unramified if λ/κ is separable, $[\lambda : \kappa] = [L:K]$

An algebraic extension is unramified if every finite subextension is unramified

Lemma: Let $L/M/K$ be a tower of extensions with L/K finite. Then L/K is unramified iff L/M, M/K are.

Pf: $\lambda : k$ is separable iff $\lambda : \mu$, $\mu : k$ are separable. Also

$$[\lambda : k] = [\lambda : \mu][\mu : k] \leq [L : M][M : K] = [L : K]$$

if $[\lambda : k] = [L : k]$ must have equality throughout
$[\lambda : k] < [L : k]$ must have inequality for one of $\lambda : \mu$, $\mu : k$.

∎

Prop: (Compositum) Inside a fixed alg. closure $\bar{K}$ let L/K be unramified. Then LM/M is unram. for all $K \subset M \subset \bar{K}$.

Cor: If $L_1, L_2 / K$ are finite unram so is $L_1 L_2 / K$

Pf: Enough to show this for L/K finite. Then $[\lambda : k]$ is finite, separable, so $\lambda = k(\bar{\alpha})$ for some $\alpha \in \mathcal{O}_L$. (primitive element thm)

(monic)

Let $f \in \mathcal{O}_K[x]$ be the min poly of α.
Then $\bar{f} \in k[x]$ is monic, $\bar{f}(\bar{\alpha}) = \overline{f(\alpha)} = 0$,
so:

$[\ell : k] = [\lambda : k] \leq \deg \bar{f} = \deg f \leq [L : K]$

so $\deg f = [L : K] = [\lambda : k]$, $\bar{f}$ is the min poly of $\bar{\alpha}$, (so irred) $L = K(\alpha)$.

Thus $LM = M(\alpha)$. Let $g \in \mathcal{O}_M[x]$ be the min poly of α over M. $(g \mid f)$

Then $\bar{g}$ (image of g in $\mu[x]$) is separable (divides $\bar{f}$). It then irred: by Hensel's Lemma any factorization would lift to $\mathcal{O}_M[x]$ since the factors would be relatively prime.

So $\bar{g}$ is the min poly of $\bar{\alpha}$ over μ

$\Rightarrow [M(\alpha) : M] = \deg g = \deg \bar{g} = [\mu(\bar{\alpha}) : \mu] \leq [\lambda' : \mu]$
$\leq [M(\alpha) : M]$

where λ' = residue field of $M(\alpha)$. So equality holds.

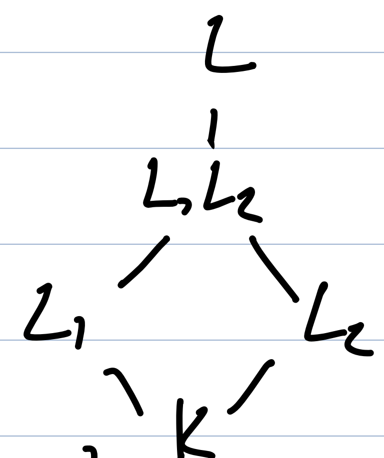
$\Rightarrow$ (1) $[LM : M] = [\lambda : \mu]$.

(2) $\lambda = \mu(\bar{\alpha})$ is separable $/\mu$. ∎

Cor: Let $L/M/K$ be a tower of algebraic extensions, then L/K is unram iff L/M, M/K are.

Pf: HW.

$\Rightarrow$ **Thm:** Let $L_1, L_2/K$ be algebraic, contained in fixed alg. ext'n L. Then L_1L_2/K is unram iff L_1/K, L_2/K are.



Cor: Let L/K be alg. Then the compositum of all unramified subextensions of K in L is an unramified subextension, necc the maximal one.

Def: Call this compositum the maximal unram subextension. In particular write K^{ur} for the max'l unramified subextension of $\bar{K}/K$.

Prop: Let T/K be the max'l unram subextension of L/K. Then the residue field τ is the separable closure of κ in λ, and L, T have same value groups

Pf: In every finite subextension of T/K, have $e=1$, so for any $\alpha \in T^\times$, $|\alpha| \in |K^\times|$ (check in $K(\alpha)$)

Next, let $\bar{\alpha} \in \lambda$ be separable over κ. Let $\bar{f}$ be the min poly of $\bar{\alpha}$, f be any monic lift to $\mathcal{O}_K[x]$. Then f is irred by Hensel's lemma,

Then $\overline{f(\alpha)} = \bar{f}(\bar{\alpha}) = 0$, $\overline{f(\alpha)'} = \bar{f}'(\bar{\alpha}) \neq 0$ since $\bar{f}$ is separable. By Hensel's Lemma, f has an actual root $\beta \in L$ lifting $\bar{\alpha}$.

Then $K(\beta)/K$ is unram so $\bar{\beta} = \bar{\alpha} \in \tau$.

Aside: $K(\alpha)/K$ should be separable for any lift α of $\bar{\alpha}$.

Recall: the value group $|K^\times|$ is the image of $K^\times$ by the absolute value: $|\cdot| : K^\times \to \mathbb{R}^\times_{>0}$

Studying a (finite) extension L / K found $L \supseteq T \supseteq K$ s.t. T/K unram, L/T is totally ramified (no unram subextension), $\lambda : \kappa$ is purely inseparable.

Ramification:

Assume K perfect, $|\cdot|_K$ has a discrete value group (so $n = ef$ in any finite extension)

Def: Say that L/K is totally ramified if it has no unramified subextension, tamely ramified if it is totally ramified and the degree of every finite subextension is prime to $p = \operatorname{char} \kappa$.

Def: $f \in \mathcal{O}_K[x]$ is an Eisenstein polynomial if: (1) it is monic; (2) $\bar{f} = x^e$, $e = \deg f$; (3) $f(0) \in \mathfrak{p}_K \setminus \mathfrak{p}_K^2$.

Prop: Let L/K be totally ramified and finite. Let $\pi \in \mathcal{O}_L$ be a uniformizer (element of $\mathfrak{p}_L \setminus \mathfrak{p}_L^2$). Then $L = K(\pi)$ and the min poly of π is Eisenstein.

Conversely, if $f \in \mathcal{O}_K$ is Eisenstein then f is irred, and if π is a root of f then $K(\pi)/K$ is totally ramified with uniformizer π.

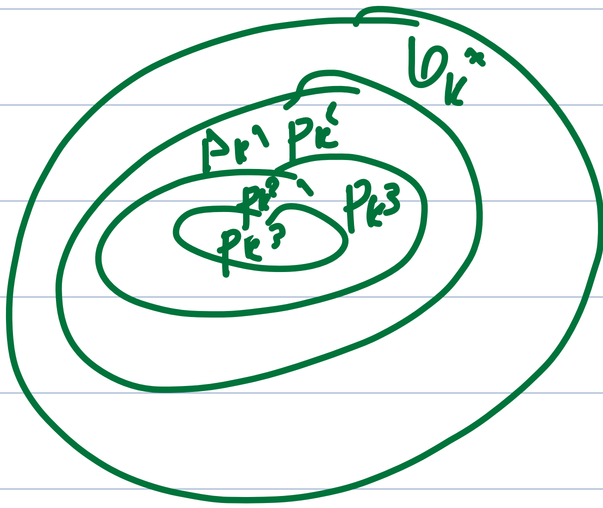
Pf: Have $e = [L:K]$, so $d = [K(\pi):K] \mid e$.
want: $d = e$.

First, all conjugates of π have same absolute value, so all coeff of f are of abs. value < 1 so in $\mathfrak{p}_K$. $\Rightarrow \bar{f} = x^d$, $d = \deg f$.

Constant coeff of f is the product of the roots so its absolute value is $|\pi|^d$.
$\Rightarrow |\pi|^d \in |K^\times|$ but $|\pi|$ generates the cyclic group $|L^\times| / |K^\times|$ which has order e. so $e \mid d$.

Since $d \mid e$, get $d = e$. $|\pi|^e$ generates the value group of K, so const coeff is in $\mathfrak{p}_K \setminus \mathfrak{p}_K^2$.

Image.



normalise valuation so $v(K^\times) = \mathbb{Z}$. Then $\mathfrak{p}_K^j \setminus \mathfrak{p}_K^{j+1} = \{x : v(x) = j\}$

Conversely, let $f \in \mathcal{O}_K[x]$ be Eisenstein of deg e, $L = K(\pi)$ where π is a root.

Write $f = \sum_{i=0}^{e} a_i x^i$ with $a_e = 1$, $a_0 \in \mathfrak{p}_K \setminus \mathfrak{p}_K^2$.
$a_i \in \mathfrak{p}_K$ if $0 < i < e$.

for $i < e$, $|a_i \pi^i| < |\pi|^i$.

If $|\pi| \geq 1$, we'd get $|a_i \pi^i| < |\pi|^e$ if $0 < i < e$
same for $|a_e| < 1 \leq |\pi|^e$.

But $0 = f(\pi) = \pi^e + \sum_{i=1}^{e} a_i \pi^i \Rightarrow \Leftarrow$

so $|\pi| < 1$, $\pi \in \mathfrak{p}_L$. For $1 \leq i \leq e-1$, $|a_i \pi^i| < |a_i| \leq |a_0|$

So from $f(\pi) = 0$ get $|\pi^e| = |a_0|$:

$$|\pi^e| = \left|\sum_{i=0}^{e-1} a_i \pi^i\right| = |a_0|$$

↑ every other summand is smaller

since $K^\times$ is generated by $|a_0|$, $e(K(\pi):K) \geq e$

so $e \leq e(K(\pi):K) \leq [K(\pi):K] = e = \deg f$

so have equality, f irred, ext'n is totally ramified,

$$|\pi| = |a_0|^{1/e}$$

so π is a uniformizer.

<u>Thm</u>: $\mathbb{Q}_p$ has finitely many extensions of any degree.

<u>Pf</u>: (HW) $\mathbb{Q}_p$ has a unique unram ext'n of any given degree. Enough to count totally ramified extensions of those fields.

Let f be an Eisenstein poly generating a field $L := T(\alpha)$, $T/\mathbb{Q}_p$ unram

Then $f'(\alpha) \neq 0$. If g is close to f, then g is also Eisenstein, $g'(\alpha)$ close to $f'(\alpha)$.

So, in a nbd of f have $\left| \frac{g(\alpha)}{(g'(\alpha))^2} \right| < 1$.

$\Rightarrow$ By Hensel's lemma g has a root in L, say β, then $L = T(\beta)$ (g Eisenstein $\Rightarrow$ irred)

So f, g determine same extension

$\left\{ \begin{matrix} \text{Eisenstein} \\ \text{poly of deg } e \end{matrix} \right\}$ is cpt. $= p_K^{e-1} \times (p_K \setminus p_K^2)$.

So only have finitely many extensions

∎

HW: if L/K totally ramified, $\exists$ max'l tamely ramified subext'n: compositum of prime-to-p subextensions. Call it T.

Then L/T is wildly ramified: every finite subext'n has deg power of p