Math 538, lecture 8 2/2/2024

Last time: Absolute values, valuations on fields

F field, $|\cdot|: F \to \mathbb{R}_{\geq 0}$ st $\begin{cases} |xy| = |x|\cdot|y| \\ |x+y| \leq |x| + |y| \\ |x| = 0 \text{ iff } x = 0 \end{cases}$

Either: (1) $|n| \leq 1$ for some $n \in \mathbb{Z}_{\geq 2}$, then $|x+y| \leq \max\{|x|, |y|\}$ — ultrametric, nonarchiemedean

(2) $\{|n| : n \in \mathbb{Z}\}$ is unbounded — archimedean case

Say $|\cdot|_1, |\cdot|_2$ are equivalent if define same topology. true iff $|x|_1 = |x|_2^\lambda$ for some $\lambda > 0$

Thm: (Ostrowski) $|\mathbb{Q}| = \{|\cdot|_p\} \cup \{|\cdot|_\infty\}$

Today: completion and complete fields

Theorem: ("Weak approximation"; Artin-Whaples) let $\{|\cdot|_i\}_{i=1}^n$ be inequivalent absolute values on F, let $\{x_i\}_{i=1}^n \subset F$, $\varepsilon > 0$. Then $\exists y \in F$ st $|y - x_i|_i < \varepsilon$.

(need $|\cdot|_i$ non-discrete)

Pf: Step 1: $\exists z_1 \in F$ st $|z_1|_1 > 1$, $|z_1|_2 < 1$.
(since $|\cdot|_i$ are nondiscrete, inequivalent)
Suppose $|z_1|_j < 1$ for all $2 \le j \le k$, $|z_1|_{k+1} \ge 1$
then $\exists w$ st $|w|_1 > 1$, $|w|_{k+1} < 1$.

If $|z_1|_{k+1} = 1$, then for all $s \in \mathbb{Z}_{\ge 1}$, $|z_1^s w|_1 > 1$
$|z_1^s w|_{k+1} < 1$
if s large enough $|z_1^s w|_j < 1$ for $2 \le j \le k$,
so replace z_1 with $z_1^s w$.

If $|z_1|_{k+1} > 1$, use instead $\frac{z_1^s w}{1+z_1^s}$

wrt $|\cdot|_j$, $2 \le j \le k$, $\frac{z_1^s w}{1+z_1^s} \xrightarrow[s\to\infty]{} 0$

wrt $|\cdot|_1, |\cdot|_{k+1}$ $\frac{z_1^s w}{1+z_1^s} = \frac{w}{1+z_1^{-s}} \xrightarrow[s\to\infty]{} w$

so for s large enough $\left|\frac{z_1^s w}{1+z_1^s}\right|_j \begin{cases} <1 & 2 \le j \le k+1 \\ >1 & j=1. \end{cases}$

Conclusion: have $\{z_i\}_{i=1}^n \subset F$ st $|z_i|_i > 1$
$|z_i|_j < 1$ if $j \ne i$.

Step 2: let $u_i = \frac{z_i^s}{\sum_{j=1}^n z_j^s}$.

Then $\sum_{i=1}^{n} u_i = 1$. As $s \to \infty$, $u_i \underset{s\to\infty}{\to} \delta_{ij}$ wrt $|\cdot|_j$

Given $\delta > 0$, take s large st $|u_i - \delta_{ij}|_j < \delta$ for all i, j

<u>Step 3:</u> Given $\underline{x} \in F^n$, let $y = \sum_i u_i x_i$

Then $|y - x_j|_j = |\sum_i (u_i - \delta_{ij}) x_i|_j$

$$\leq \sum_i |u_i - \delta_{ij}|_j \cdot |x_i|_j$$

$$\leq \sum_i \delta \cdot |x_i|_j < \varepsilon$$

if δ is small enough $\blacksquare$

Completions

<u>Lemma:</u> Let (X, d_X), (Y, d_Y) be two metric spaces, let $f: X \to Y$ is uniformly continuous on balls. Then there is a unique continuous function $\hat{f}: \hat{X} \to \hat{Y}$ s.t. $\hat{f}\restriction_X = f$

(write $\hat{X}$ = completion of X wrt d_X)

<u>Cor:</u> Let $|\cdot|$ be an absolute value on F. Then the field operations and $|\cdot|$ extend to continuous function on $\hat{F}$, giving it the structure of a complete field.

<u>Pf:</u> $+$ is uniformly cts:

$$|(x+y)-(z+w)| \le |x-z| + |y-w|$$

$\cdot$:
$$|xy - zw| \le |(x-z)y| + |z(y-w)|$$
$$\le R\left(|x-z| + |y-w|\right)$$

if $y, z \in B(R)$.

<u>divisions:</u> if $\{x_n\} \subset F$ is Cauchy, $x_n \not\to 0$, then $|x_n|$ are eventually bounded below, say $|x_n| > \varepsilon$, then

$$\left|\frac{1}{x_n} - \frac{1}{x_m}\right| = \left|\frac{x_n - x_m}{x_n x_m}\right| < \varepsilon^{-2} |x_n - x_m|$$

so $\{\frac{1}{x_n}\}$ also Cauchy, inverse to $\{x_n\}$ in $\hat{F}$.

∎

Notation: usually write $|\cdot|_v$ for some abs value and F_v for corresponding completion

Example: The completions of $\mathbb{Q}$ are $\mathbb{R} = \mathbb{Q}_\infty$, and $\mathbb{Q}_p$ for $2 \le p < \infty$

field of p-adic numbers

Lemma: let F be a field complete wrt a non-arch absolute value. Then $\sum_{n=1}^{\infty} a_n$ converges in F iff $a_n \xrightarrow[n\to\infty]{} 0$.

Pf: Exercise

Def: $\mathbb{Z}_p = \{ x \in \mathbb{Q}_p \mid |x|_p \le 1 \}$ is the ring of p-adic integer

Lemma: (1) $\mathbb{Z}_p \subset \mathbb{Q}_p$ is an open ($\Rightarrow$ closed) subring.

(2) $\mathbb{Z}$ is dense in $\mathbb{Z}_p$

(3) the map $\mathbb{Z}_p / p^k \mathbb{Z}_p \leftarrow \mathbb{Z}/p^k\mathbb{Z}$ is an isom

(4) Every element of $\mathbb{Z}_p$ has a unique representation in the form $\sum_{j \ge 0}^{\infty} a_j p^j$

$a_j \in \{0, .., p-1\}$

(5) $\mathbb{Z}_p$ is compact.

Pf: (1) For any ultrametric absolute value, $|1| \le 1$
If $|x|, |y| \le 1$ then $|xy| = |x||y| \le 1$
$$|x \pm y| \le \max\{|x|, |y|\} \le 1$$
so $\{x : |x| \le 1\}$ is a ring.

recall $v_p(p^k \frac{a}{b}) = k$, $|p^k \frac{a}{b}| = p^{-k}$ if $p \nmid ab$,
so the value group of $|\cdot|$ is $\{p^k\}_{k \in \mathbb{Z}}$
which is discrete in $\mathbb{R}^{\times}_{>0}$

$\Rightarrow$ elements of $\mathbb{Q}_p^{\times}$ also have absolute values in $p^{\mathbb{Z}}$, so
$$B_{\mathbb{Q}_p}(0,1) = B_{\mathbb{Q}_p}(0, p/2)$$

so $B_{\mathbb{Q}_p}(0,1) = B^{o}_{\mathbb{Q}_p}(0, 1+\epsilon)$, so is open

In fact, for any ultrametric, closed balls are open:
$$\text{If } y \in B(x,r), \text{ then } B(y,r) \subseteq B(x,r).$$

(2) Given $x \in \mathbb{Z}_p$, $r \ge 0$, have $p^k \frac{a}{b} \in \mathbb{Q}$ st. $p \nmid ab$
$$|p^k \tfrac{a}{b} - x|_p < p^{-r}.$$

then $p^k \frac{a}{b} - x \in \mathbb{Z}_p$ so $p^k \frac{a}{b} \in \mathbb{Z}_p$, so $|p^k| \leq 1$

so $k \geq 0$.

Since $p \nmid b$, have $\underset{\mathbb{Z}}{\bar{b}}$ s.t. $b \cdot \bar{b} \equiv 1 \ (p^r)$

Then $p^k a \bar{b} \in \mathbb{Z}$, $|p^k a \bar{b} - p^k \frac{a}{b}|_p$

$= |p^k|_p \cdot |a|_p \cdot |\frac{1}{b}|_p \cdot |b\bar{b} - 1|_p$

$\leq p^{-k} \cdot 1 \cdot 1 \cdot p^{-r} \leftarrow p^r | (b\bar{b} - 1)$

$\leq p^{-r}$

so $|p^k a \bar{b} - x|_p \leq \max \left\{ |p^k a \bar{b} - p^k \frac{a}{b}|_p, \ |p^k \frac{a}{b} - x|_p \right\}$

$\leq p^{-r}$.

(3) Know $\mathbb{Z}$ dense in $\mathbb{Z}_p$, $\underset{B(0, p^{-k})}{\underset{\shortparallel}{p^k \mathbb{Z}_p}}$ is open

so $\mathbb{Z} + p^k \mathbb{Z}_p = \mathbb{Z}_p$

$\Rightarrow \mathbb{Z}$ surjects on $\mathbb{Z}_p / p^k \mathbb{Z}_p$.

Now $\mathbb{Z} \cap p^k \mathbb{Z}_p = \{ x \in \mathbb{Z} : |x|_p \leq p^{-k} \} = p^k \mathbb{Z}$

Get isom $\mathbb{Z}/p^k\mathbb{Z} \to \mathbb{Z}_p/p^k\mathbb{Z}_p$

($p^k\mathbb{Z}_p$ is rescaling of ball $B(0,1)$ by p^k
so is the ball of radius $|p^k|_p = 1/p^k$.)
(Easy to see $B(0,\varepsilon) \subset B(0,1)$ is an ideal for all ε)

$\Rightarrow$ Picture: $\mathbb{Z}_p = \coprod \mathbb{Z}_p/p^k\mathbb{Z}_p$

$$= \coprod_{\substack{a \bmod p^k \\ a \in \mathbb{Z}}} a + p^k\mathbb{Z}_p$$

Write $\mathbb{Z}_p = B(0,1)$ as the union of p balls of radius $\frac{1}{p}$, each of which is a union of p balls of rad $\frac{1}{p^2}$, ...

Cor: $\mathbb{Z}_p$ is totally disconnected. (ctd components are points)
[But $\mathbb{Z}_p$ is not discrete!]

Viewed $\mathbb{Z}_p$ as a multiscale arrangement of balls. Elements of $\mathbb{Z}_p \leftrightarrow$ infinite paths in the tree of balls

Conversely, any path in the tree is a point: Such a path is a sequence of residue classes $b_1 \bmod p$, $b_2 \bmod p^2$, ..., $b_k \bmod p^k$, .

$$b_k \equiv b_{k+1} \bmod p^k$$

$\Rightarrow \{b_k\}$ is a Cauchy sequence in $\mathbb{Z}$ wrt $|\cdot|_p$
$\Rightarrow$ limit exists in $\mathbb{Z}_p$.

$$\Rightarrow \mathbb{Z}_p = \{ (b_k)_{k\geq 1} \in \prod_{k\geq 1} (\mathbb{Z}/p^k\mathbb{Z}) \mid b_\ell \equiv b_k \ (p^k) \text{ if } \ell \geq k \}$$

(not obvious what $\mathbb{Z}$ is from this pov)

(4) Let $A \subset \mathbb{Z}$ be a set of representatives for $\mathbb{Z}/p\mathbb{Z}$.

Let

$$f : A^{\mathbb{N}} \to \mathbb{Z}_p \quad \text{be} \quad f(\underline{a}) = \sum_{j\geq 0}^{\infty} a_j p^j.$$

Then $|a_j p^j|_p \leq p^{-j} \xrightarrow[j\to\infty]{} 0$ so series converges and f is well-defined in $\mathbb{Q}_p$. But partial sums valued in $\mathbb{Z} \subset \mathbb{Z}_p$, $\mathbb{Z}_p$ is closed.

f is cts if $\underline{a}, \underline{a}'$ agree at first k co-ords then $f(\underline{a}) - f(\underline{a}') = \sum_{j \geq k}^{\infty} (a_j - a_j') p^j$

$$\text{so } |f(\underline{a}) - f(\underline{a}')|_p \leq p^{-k}$$

if $a_k \neq a_k'$ then $f(\underline{a}) - f(\underline{a}') = (a_k - a_k') p^k$
$$+ \sum_{j > k} (a_j - a_j') p^j$$

$|a_k - a_k'| \geq 1$ (not divisible by p)

so $|(a_k - a_k') p^k|_p = p^{-k}$, tail has $|\cdot|_p \leq p^{-k-1}$.

so $|f(\underline{a}) - f(\underline{a}')|_p = p^{-k}$, k = first time where $a_k \neq a_k'$.

$\Rightarrow f^{-1}$ also cts.

Surjectivity of $f \Rightarrow$ (4), (5)

given $(b_k)_{k \geq 1}^{\infty}$, b_k mod p^k.

say chose $\{a_j\}_{j=0}^{k-1}$ s.t. $\sum_{j=0}^{k-1} a_j p^j \equiv b_k \ (p^k)$?

then $b_{k+1} - \sum_{j=0}^{k-1} a_j p^j$ divisible by p^k

divide by p^k $\Rightarrow$ $\exists a_k$ s.t. $b_{k+1} - \sum_{j=0}^{k-1} a_j p^j \equiv a_k p^k$ mod p^{k+1}.

Or: enough to represent every $n \in \mathbb{Z}$.

⊙ for (ii): for every k can cover $\mathbb{Z}_p$ by p^k balls of radius p^{-k}.

Cor: $\mathbb{Z}_p$ is a maximal compact subring of $\mathbb{Q}_p$; topology of $\mathbb{Q}_p$ is generated by the balls $p^r \mathbb{Z}_p$

$\Rightarrow$ $\mathbb{Q}_p$ is locally compact.