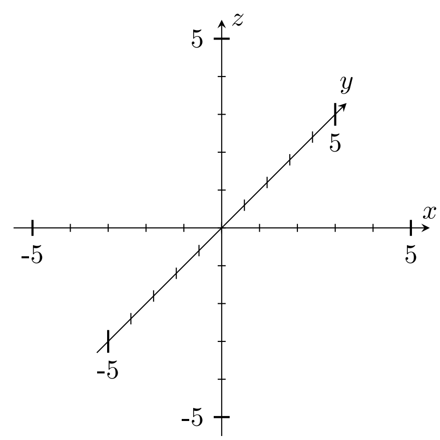
Math 100:V02 – WORKSHEET 17
MULTIVARIABLE DIFFERENTIATION

1. Plotting in three dimensions

(1) ⋆ Plot the points $(2, 1, 3)$, $(-2, 2, 2)$ on the axes provided.



(2) Let $f(x, y) = e^{x^2+y^2}$.

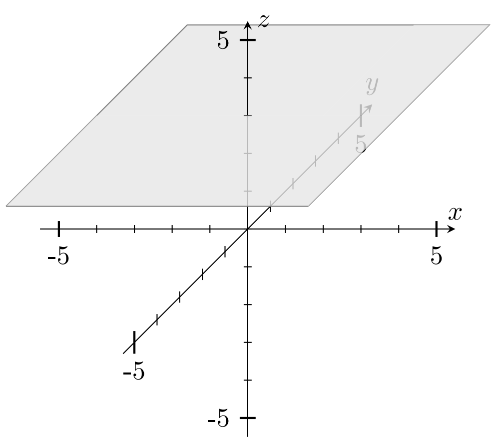
(a) ⋆ What are $f(0, -1)$? $f(1, 2)$? Plot the point $(0, 1, f(0, 1))$ on the axes provided.

(b) ⋆ What is the *domain* of f (that is: for what (x, y) values does f make sense?

(c) ⋆ What is the *range* of f (that is: what values does it take)?

(3) ⋆⋆ What would the graph of $z = \sqrt{1 - x^2 - y^2}$ look like?

(4) ⋆ Which plane is this?



(A) $x = 3$
(B) $y = 3$
(C) $z = 3$
(D) none
(E) not sure

Date: 4/4/2024, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

2. Partial derivatives

(5) (a) $\star$ Let $f(x) = 2x^2 - a^2 - 2$. What is $\frac{df}{dx}$?

(b) $\star$ Let $f(x) = 2x^2 - y^2 - 2$ where y is a constant. What is $\frac{df}{dx}$?

(c) $\star$ Let $f(x, y) = 2x^2 - y^2 - 2$. What is the rate of change of f as a function of x *if we keep y constant*?

(d) $\star$ What is $\frac{\partial f}{\partial y}$?

(6) Find the partial derivatives with respect to both x, y of

(a) $\star$ $g(x, y) = 3y^2 \sin(x + 3)$

(b) $\star$ $h(x, y) = ye^{Axy} + B$

(7) The the gravitational *potential* due to a point mass M (equivalently the electrical potential due to a point charge M) is given by the formula $U(x, y, z) = -\frac{GM}{r}$ where $r = \sqrt{x^2 + y^2 + z^2}$. Here G is the universal gravitational constant (equivalently G is the Coulomb constant).

(a) $\star$ The x-component of the field is given by the formula $F_x(x, y, z) = -\frac{\partial U}{\partial x}$. Find F_x

(b) $\star$ The magnitude of the field is given by $\left|\vec{F}\right| = \sqrt{F_x^2 + F_y^2 + F_z^2}$. How does it decay as a function of r?

(8) The *entropy* of an ideal gas of N molecules at temperature T and volume V is

$$S(N, V, T) = Nk \log \left[\frac{VT^{1/(\gamma-1)}}{N\Phi} \right].$$

where k is *Boltzmann's constant* and γ, Φ are constants that depend on the gas.

(a) $\star$ Find the *heat capacity at constant volume* $C_V = T\frac{\partial S}{\partial T}$.

(b) $\star\star\star$ Using the relation ("ideal gas law") $PV = NkT$ write S as a function of N, P, T instead. Differentiating with respect to T **while keeping** P **constant** determine the *heat capacity at constant pressure* $C_P = T\frac{\partial S}{\partial T}$.

Notations for the partial derivative include $\frac{\partial f}{\partial x}$, $\frac{\partial}{\partial x} f$, $\partial_x f$, $D_x f, f_x$.

(9) We can also compute second derivatives. For example $f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2}{\partial y \partial x} f$. Evaluate:

(a) $\star$ $h_{xx} = \frac{\partial^2 h}{\partial x^2} =$

(b) $\star$ $h_{xy} = \frac{\partial^2 h}{\partial y \partial x} =$

(c) $\star$ $h_{yx} = \frac{\partial^2 h}{\partial x \partial y} =$

(d) $\star$ $h_{yy} = \frac{\partial^2 h}{\partial y^2} =$

(10) $\star$ Repeat this exercise for the function g from problem 2(a).

(11) You stand in the middle of a north-south street (say Health Sciences Mall). Let the x axis run along the street (say oriented toward the south), and let the y axis run across the street. Let $z = z(x, y)$ denote the height of the street surface above sea level.

(a) $\star$ What does $\frac{\partial z}{\partial y} = 0$ say about the street?

(b) $\star$ What does $\frac{\partial z}{\partial x} = 0.15$ say about the street?

(c) $\star$ You want to follow the street downhill. Which way should you go?

(d) The intersection of Health Sciences Mall and Agronomy Road is a local maximum. What does that say about $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ there?

3. Bonus (<u>**nonexaminable!**</u>): multivariable linear and higher approximation

Definition 1. A function $f(x, y)$ is *differentiable* at x_0, y_0 if we have a linear approximation $f(x, y) = f(x_0, y_0) + A(x - x_0) + B(y - y_0) + \text{small}$ as $(x, y) \to (x_0, y_0)$. We then have $A = \frac{\partial f}{\partial x}(x_0, y_0)$ and $B = \frac{\partial f}{\partial y}(x_0, y_0)$. The definition for functions of more than two variables is analogous.

(12) Let $f(x) = \sqrt{2 + x^2 + y^2}$.

(a) Write the linear approximation to f about $(1, 1)$ and use that to estimate $f(1.1, 1.2)$.

(b) Write the linear approximation to f about $(3, 5)$ and use that to estimate $f(2.8, 4.9)$.