Math 100:V02 – SOLUTIONS TO WORKSHEET 6
EXPONENTIAL AND TRIG FUNCTIONS

1. Review: Arithmetic of derivatives

(1) Differentiate
   (a) (Final, 2016) $g(x) = x^2 e^x$ (and then also $x^a e^x$)
   Solution: Applying the product rule we get $\frac{dg}{dx} = \frac{d(x^2)}{dx} \cdot e^x + x^2 \cdot \frac{d(e^x)}{dx} = (2x + x^2)e^x = x(x+2)e^x$, and in general
$$\frac{d}{dx}(x^a e^x) = ax^{a-1}e^x + x^a e^x = x^{a-1}(x+a)e^x\,.$$
   (b) (Final, 2016) $h(x) = \frac{x^2+3}{2x-1}$
   Solution: Applying the quotient rule the derivative is $\frac{2x\cdot(2x-1)-(x^2+3)\cdot 2}{(2x-1)^2} = \frac{4x^2-2x-2x^2-6}{(2x-1)^2} = 2\frac{x^2-x-3}{(2x-1)^2}$.

(2) Let $f(x) = \frac{x}{\sqrt{x}+A}$. Given that $f'(4) = \frac{3}{16}$, give a quadratic equation for A.
   Solution: $f'(x) = \frac{1\cdot\left(\sqrt{x}+A\right) - x\left(\frac{1}{2}x^{-1/2}\right)}{\left(\sqrt{x}+A\right)^2} = \frac{\sqrt{x}+A-\frac{1}{2}\sqrt{x}}{\left(\sqrt{x}+A\right)^2} = \frac{\frac{1}{2}\sqrt{x}+A}{\left(\sqrt{x}+A\right)^2}$. Plugging in $x = 4$ we have
$$\frac{3}{16} = f'(4) = \frac{1+A}{(2+A)^2}$$
so we have
$$3(2+A)^2 = 16(1+A)$$
that is
$$3A^2 + 12A + 12 = 16 + 16A$$
that is
$$3A^2 - 4A - 4 = 0\,.$$
In fact this gives $A = -\frac{2}{3}, 2$.

(3) Suppose that $f(1) = 1$, $g(1) = 2$, $f'(1) = 3$, $g'(1) = 4$.
   (a) What are the linear approximations to f and g at $x = 1$? Use them to find the linear approximation to fg at $x = 1$.
   Solution: We have
$$f(x) \approx f(1) + f'(1)(x-1) = 1 + 3(x-1)$$
$$g(x) \approx g(1) + g'(1)(x-1) = 2 + 4(x-1)$$
   multiplying them we have
$$\begin{aligned}(fg)(x) &\approx (1+3(x-1))(2+4(x-1)) \\ &= 2 + 1\cdot 4(x-1) + 2\cdot 3(x-1) + 12(x-1)^2 \\ &\approx 2 + 10(x-1)\end{aligned}$$
   to first order.
   (b) Find $(fg)'(1)$ and $\left(\frac{f}{g}\right)'(1)$.
   Solution: $(fg)'(1) = f'(1)g(1) + f(1)g'(1) = 3\cdot 2 + 1\cdot 4 = 10$.
$$\left(\frac{f}{g}\right)'(1) = \frac{f'(1)g(1) - f(1)g'(1)}{(g(1))^2} = \frac{3\cdot 2 - 1\cdot 4}{2^2} = \frac{1}{2}\,.$$

(4) Evaluate

Date: 25/1/2024, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

(a) $(x\cdot x)'$ and $(x')\cdot(x')$. What did we learn?
Solution: $(x\cdot x)' = (x^2)' = 2x$ while $(x')\cdot(x') = 1\cdot 1 = 1$ – the "rule" $(fg)' = f'g'$ is **wrong**.

(b) $\left(\frac{x}{x}\right)'$ and $\frac{(x')}{(x')}$. What did we learn?
Solution: $\left(\frac{x}{x}\right)' = (1)' = 0$ while $\frac{(x')}{(x')} = \frac{1}{1} = 1$ – the "rule" $\left(\frac{f}{g}\right)' = \frac{f'}{g'}$ is **wrong**.

2. Exponentials

(5) Simplify

(a) $\left(e^5\right)^3$, $\left(2^{1/3}\right)^{12}$, 7^{3-5}.
Solution: $\left(e^5\right)^3 = e^{5\cdot 3} = e^{15}$, $\left(2^{1/3}\right)^{12} = 2^{\frac{1}{3}\cdot 12} = 2^4 = 16$, $7^{3-5} = 7^{-2} = \frac{1}{49}$.

(b) $\log(10e^5)$, $\log(3^7)$.
Solution: $\log\left(10e^5\right) = \log(10) + 5\log(e) = \log(10) + 5$, $\log(3^7) = 7\log 3$.

(6) Differentiate:

(a) 10^x
Solution: This is $(\log 10)\cdot 10^x$.

(b) $\frac{5\cdot 10^x + x^2}{3^x+1}$
Solution: By the quotient rule this is

$$\frac{(5\log 10\cdot 10^x + 2x)(3^x+1) - \left(5\cdot 10^x + x^2\right)\log 3\cdot 3^x}{(3^x+1)^2}.$$

3. Trigonometric functions

(7) (Special values) What is $\sin\frac{\pi}{3}$? What is $\cos\frac{5\pi}{2}$?
Solution: $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$, $\cos\left(\frac{5\pi}{2}\right) = \cos\left(\frac{\pi}{2} + 2\pi\right) = \cos\left(\frac{\pi}{2}\right) = 0$.

(8) Derivatives of trig functions

(a) Interpret $\lim_{h\to 0}\frac{\sin h}{h}$ as a derivative and find its value.
Solution: This is $\lim_{h\to 0}\frac{\sin(0+h)-\sin 0}{h} = \frac{d\sin x}{dx}\Big|_{x=0} = \cos 0 = 1$.

(b) Differentiate $\tan\theta = \frac{\sin\theta}{\cos\theta}$.
Solution: Applying the quotient rule we get

$$\frac{\mathrm{d}\tan\theta}{\mathrm{d}\theta} = \frac{\cos\theta\cdot\cos\theta - \sin\theta\cdot(-\cos\theta)}{\cos^2\theta}$$
$$= \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}.$$

We also have

$$\frac{\mathrm{d}\tan\theta}{\mathrm{d}\theta} = \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta} = 1 + \tan^2\theta$$

which is sometimes useful.

(9) What is the equation of the line tangent the graph $y = T\sin x + \cos x$ at the point where $x = \frac{\pi}{4}$?
Solution: We have $y(\frac{\pi}{4}) = \frac{T}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{T+1}{\sqrt{2}}$. Also, $\frac{\mathrm{d}y}{\mathrm{d}x} = T\cos x - \sin x$ so $\frac{\mathrm{d}y}{\mathrm{d}x}|_{x=\frac{\pi}{4}} = \frac{T}{\sqrt{2}} - \frac{1}{\sqrt{2}} = \frac{T-1}{\sqrt{2}}$. So the line is

$$y = \frac{T-1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right) + \frac{T+1}{\sqrt{2}}.$$