Math 100, lecture 21, 26/3/2024

Last time: more on Euler's method

Today: (1) Basic skills

(2) Checking your work

Math 100:V02, Winter Term 2024
Worksheet 16

March 26th, 2024

Instructions

- Find all errors

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1. Differentiate $f(x) = 3x^3 + \frac{7}{x^{3/2}}$.

$$f'(x) = \cancel{3}^{\,9}x^2 + \frac{\cancel{1}^{\,0}\cdot x^{3/2} - 7\cdot\frac{\overset{3}{1}}{2}x^{1/2}}{(x^{3/2})^2}$$

(better: $\frac{7}{x^{3/2}} = 7x^{-3/2}$)

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2. Evaluate $\lim_{t\to3} \frac{(t-3)\sin t}{t^2-9}$

$$\frac{(t-3)\sin t}{t^2 9} = \frac{0}{0} \not= \infty$$

← 0/0 not a number

undef

fcn of t

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3. Determine the asymptotics as $x \to \infty$ **of** $f(x) = \frac{\sqrt{x^4+ax}-1}{bx+c}$ **if** a, b, c **are nonzero constants.**

outside $\sqrt{\ }$

$$\frac{\sqrt{x^4+ax-1}}{bx+c} \sim \frac{\cancel{ax-1}^{\sqrt{x^4}}}{bx} \sim \frac{\cancel{1}}{\cancel{bx}}$$

$\frac{a}{b}$ (if $\frac{ax-1}{bx}$)

$\frac{x}{b}$ (right answer)

4. Differentiate $g(x) = \frac{e^{\sin x}}{x^2}$.

chain rule

$$g'(x) = -2\frac{e^{\sin x}}{x^3} + \frac{e^{\sin x}}{x^2}\cos x$$

$(x^{-2})' = -2x^{-3}$

$64^{\frac{2}{3}} = \sqrt[3]{64}$

5. Approximate $\sqrt[3]{7}$ using a linear approximation.

Let $h(x) = x^{1/3}$ Then $h'(x) = \frac{1}{3}x^{-2/3}$ so $h'(8) = \frac{8^{-2/3}}{3} = \frac{64^{-1/3}}{3} = \cancel{2\tfrac{1}{3}}\ \frac{1}{4} \cdot \frac{1}{3} = \frac{1}{12}$

so $h(7) \approx h(8) + \cancel{2}\tfrac{1}{3} \cdot (\cancel{8-7}) = \cancel{2\tfrac{1}{3}}\ 2 - \frac{1}{12} = \frac{23}{12}$

$7-8$

$$\boxed{32^{-4/5} = (32^{\frac{1}{5}})^{-4} = 2^{-4} = 1/16}$$

6. Find the line tangent to the curve $3x^2y + y^3 = (x+y)^2$ at the point $(1,1)$.

$3x^2y' + 6xy + 3y^2y' = 2(x+y)(1+y')$ ← chain rule

so $3y' + 6 + 3 = 4$ so $y' = -2$ so the line is $y = -2(x-1)+1$

7. Differentiate the function $x \log x$ with respect to x.

$(x\log x)' = \log x + \frac{x}{\cancel{\log x}}\ x$

8. Let f be a function such that $f'(x) = \frac{(x-3)(x+5)}{x^4+1}$. Find the regions where f is increasing and decreasing.

Increase $(\cancel{-3}\ 3, \infty) \cup (-\infty, -5)$

Decrease ~~$(-\infty, -3)$~~ $(-5, 3)$

$x+5$ changes sign

$x-3$ " "

at 3

not -3

9. The volume V of an expanding spherical baloon of radius r is given by $V = \frac{4}{3}\pi r^3$. At the moment when $r = 3\text{cm}$ we have $\frac{dr}{dt} = \frac{1}{\pi}\frac{\text{cm}}{\text{sec}}$. How fast is the volume changing at that moment?

$V' = 4\pi r^2 = 36\pi$ ← no units

needed $\frac{dV}{dt} = \frac{dV}{dr}\cdot\frac{dr}{dt} = 36\pi\cdot\frac{1}{\pi} = 36\,\frac{\text{cm}}{\text{sec}}$

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10. Find the second order Taylor polynomial of $f(x) = e^{x^2} + x^3$ about $x = 0$.

$f'(x) = 2xe^{x^2} + 3x^2$

$f''(x) = 2e^{x^2} + 4x^2e^{x^2} + \overset{6}{\cancel{2}}x$

$T_2(x) = \overset{f(0)}{(\cancel{e^{x^2} + x^3})} + \overset{f'(0)}{(\cancel{2xe^{x^2} + x^2})}\cdot x + \frac{1}{2}\underset{f''(0)}{(\cancel{2e^{x^2} + 4x^2e^{x^2} + 2x})}\,x$ X

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11. Suppose the function f has $f(x) \approx 5 + 2(x-3) + (x-3)^3$ to third order about $x = 3$. What is $f''(3)$?

$T_3''(x)$ $f''(\cancel{x}) = 6(x-3) \Rightarrow T_3''(3) = 0$

$=$

$f''(3)$

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12. Find $\lim_{x\to 1}\frac{\tan x}{(x-1)^2}$

$\frac{\tan 1}{0}$

13. Suppose that $f'(3) = 8$. Find the derivative of $f\left(x^2+2\right)$ at $x = 1$.

✓ $f'(1+2) = 8$

$\frac{d}{dx} f(x^2+2) = f'(x^2+2) \cdot 2x$

forgot chain rule

so $\left.\frac{d}{dx}\right|_{x=1} f(x^2+1) = f'(3) \cdot 2 \cdot 1 = 16$

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14. Differentiate $\frac{x^2}{x+a}$

$\frac{2x}{(x+a)^2}$

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15. Determine the asymptotics of $g(x) = \frac{x^7+5\sin x+e^{3x}}{x^5+3}$ as $x \to \infty$.

$$\frac{x^7 + 5\sin x + e^{3x}}{x^5+3} \sim \frac{x^7}{x^5} = \infty$$

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16. Find the fourth order Taylor polynomial of $f(x) = \frac{e^{x^2}}{1+x^2}$ about $x = 0$.

$$e^{x^2} \sim 1 + x^2 + \frac{x^4}{2}$$

$$\frac{1}{1+x^2} \sim 1 + x^2 + x^4$$