Math 100, Lecture 3

Last time: Asymptotics of expressions.

Idea: If (in some limit) $f \ll g$ then $f+g \sim g$.

Example: As $x \to \infty$, $1 - x^2 + x^4 \sim x^4$

near 0 (as $x \to 0$) $1 - x^2 + x^4 \sim 1$

(to second order in x, $1 - x^2 + x^4 \approx 1 - x^2$)

("leading order" = most important term)
("second order" = vanishing like x^2 at most)

Today: (1) If $f \sim \bar{f}$, $g \sim \bar{g}$ then $fg \sim \bar{f}\bar{g}$

$\frac{f}{g} \sim \frac{\bar{f}}{\bar{g}}$

(2) Asymptotics at $a \neq 0, \infty$

(3) Limits

Tutorials
APSC: Thu 11-12, ORCH 4062
Science: Tue 14-15, ORCH 3062

Math 100:V02 – WORKSHEET 2
LIMITS

1. Asymptotics

(1) How does the each expression behave when x is large? small? what is x is large but negative? Sketch a plot

(a) $ax^3 - bx^5$ $(a, b > 0)$

(b) $e^x - x^4$

As $x \to \infty$ $\quad e^x - x^4 \sim e^x$

As $x \to 0$ $\quad e^x - x^4 \sim 1$ $\quad (e^x \sim 1,\ x^4 \sim x^4)$

As $x \to -\infty$ $\quad e^x - x^4 \sim -x^4$ $\quad$ (e^x is decaying)

<u>remarks</u> $\lim_{x \to \infty} e^x - x^4 = \lim_{x \to \infty} e^x = \infty$

Date: 16/1/2024, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

(2) Say each expression in words, and then determine its asymptotics near 0 and near ∞.

(a) $e^{|x-5|^3}$

The exponential of the cube of the absolute value of x minus 5.

Near 0, $e^{|x-5|^3} \sim e^{125}$

Near ∞, $x-5 \sim x$, $|x-5| \sim x$, $|x-5|^3 \sim x^3$

$$e^{|x-5|^3} \underset{\substack{\uparrow \\ x \text{ large}}}{=} e^{x^3-15x^2+75x-125} = e^{x^3} \cdot e^{-15x^2} \cdot e^{75x} \cdot e^{-125}$$

<u>warning</u>: <u>not</u> e^{x^3}. (all factor matter when multiplying)

(b) $\frac{1+x}{1+2x-x^2}$

As $x \to \infty$ $\quad 1+x \sim x$, $\quad 1+2x-x^2 \sim -x^2$ $\Rightarrow \frac{1+x}{1+2x-x^2} \sim \frac{x}{-x^2} \sim -\frac{1}{x}$

$\left[\text{Or: } \frac{1+x}{1+2x-x^2} \sim \frac{x}{-x^2} \sim -\frac{1}{x}\right]$

As $x \to 0$ $\quad \frac{1+x}{1+2x-x^2} \sim \frac{1}{1} = 1$

(c) $\frac{e^x+A\sin x}{e^x-x^2}$

As $x\to\infty$, $\frac{e^x+A\sin x}{e^x-x^2} \sim \frac{e^x}{e^x} \sim 1$

As $x\to 0$ $\frac{e^x+A\sin x}{e^x-x^2} \sim \frac{1}{1} \sim 1$

As $x\to -\infty$ (tricky) : $e^x+A\sin x$ sometimes $e^x < A\sin x$

$e^x-x^2 \sim -x^2$,

sometimes $e^x > A\sin x$

(as $x\to-\infty$)

So $\frac{e^x+A\sin x}{e^x-x^2} \sim -\frac{e^x+A\sin x}{x^2}$ (no simpler description)

(d) $\frac{Ae^{rt}+Be^{-st}}{t+t^2}$ where $r, s > 0$ and $A, B \neq 0$.

As $t\to\infty$ $\frac{Ae^{rt}+Be^{-st}}{t+t^2} \sim \frac{Ae^{rt}}{t^2}$

As $t\to-\infty$ $\frac{Ae^{rt}+Be^{-st}}{t+t^2} \sim \frac{Be^{-st}}{t^2}$

As $t\to 0$ $\frac{Ae^{rt}+Be^{-st}}{t+t^2} \sim \frac{A+B}{t}$ $(e^{rt}\sim 1)$

$\left(\lim_{t\to\infty} (\text{expression}) = \lim_{t\to\infty} \frac{Ae^{rt}}{t^2} = \infty\right.$ $\left.(e^{rt} \text{ beats } t^2)\right)$

(3) Find the asymptotics of the indicated expression at the given point.

(a) $\frac{x^5+Ax^3+x}{Bx^4-x^2}$ as $x \to 0$.

(b) $\frac{x^2+1}{x-4}$ as $x \to 3$.

(c) $f(x) = \frac{x^2+1}{x-4}$ as $x \to 4$.

Small parameter $x-4$

$$\frac{x^2+1}{x-4} \sim \frac{17}{x-4}$$

(d) $f(x) = x^2 - 1$ as $x \to 1$.

As $x \to 1$, $x^2-1 = (x-1)(x+1) \sim 2(x-1)$ $x \to 1$

Or: $x^2-1 = ((x-1)+1)^2-1 = (x-1)^2+2(x-1)+1-1$

$= 2(x-1) + (x-1)^2 \sim 2(x-1)$

rebasing x^2-1 at 1

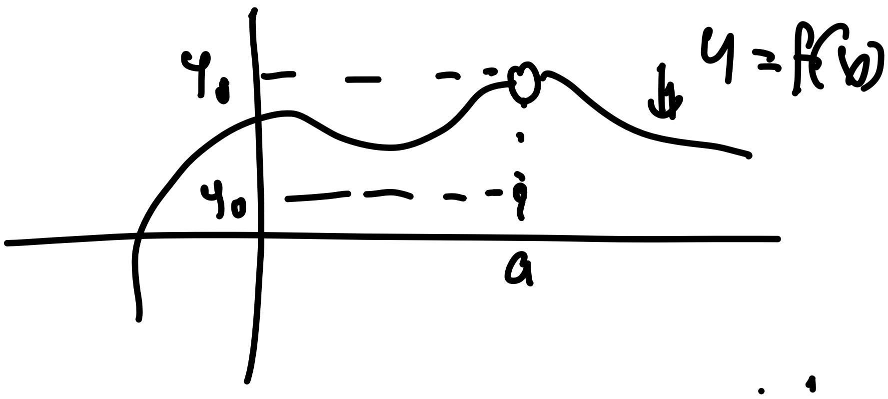
as $x-1 \to 0$, $(\text{small})^2 \ll (\text{small})$

Or: let $u = x-1$ then $x = u+1$ so   as $u \to 0$

$x^2-1 = (u+1)^2-1 = u^2+2u+1-1 = 2u+u^2 \sim 2u = 2(x-1)$

Limit: The value the function "would like" to have.

Independent of actual value



$\lim_{x \to a} f(x) = y_0$; $f(a) = y_0$

Promise: If f is defined by formula at & near a, then $\lim_{x \to a} f(x) = f(a)$

2. Limits

(4) Either evaluate the limit or explain why it does not exist. Sketching a graph might be helpful.

(a) $\lim_{x\to 5}\left(x^3 - x\right)$ $= 5^3 - 5 = 125 - 5 = 120$

(b) $\lim_{x\to 1} f(x)$ where $f(x) = \begin{cases} \sqrt{x} & 0 \le x < 1 \\ 3 & x = 1 \\ 2 - x^2 & x > 1 \end{cases}$.

$$\left.\begin{array}{l} \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \sqrt{x} = \sqrt{1} = 1 \\ \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 2-x^2 = 2-1^2 = 1 \end{array}\right\} \Rightarrow \lim_{x\to 1} f(x) = 1$$

(c) $\lim_{x\to 1} f(x)$ where $f(x) = \begin{cases} \sqrt{x} & 0 \le x < 1 \\ 1 & x = 1 \\ 4 - x^2 & x > 1 \end{cases}$.

(5) Let $f(x) = \frac{x-3}{x^2+x-12}$.

(a) (Final 2014) What is $\lim_{x\to 3} f(x)$?

(b) What about $\lim_{x\to -4} f(x)$?

(6) Evaluate

(a) $\lim_{x\to\infty} \frac{e^x + A\sin x}{e^x - x^2}$

(b) $\lim_{x\to 0} \frac{e^x + A\sin x}{e^x - x^2}$

(c) $\lim_{x\to -\infty} \frac{e^x + A\sin x}{e^x - x^2}$

(7) Evaluate

(a) $\lim_{x\to 2} \frac{x+1}{4x^2-1}$

(b) (Final, 2014) $\lim_{x\to -3^+} \frac{x+2}{x+3}$.

(c) $\lim_{x\to 1} \frac{e^x(x-1)}{x^2+x-2}$

(d) $\lim_{x\to -2^-} \frac{e^x(x-1)}{x^2+x-2}$

(e) $\lim_{x\to 1} \frac{1}{(x-1)^2}$

(f) $\lim_{x\to 4} \frac{\sin x}{|x-4|}$

(g) $\lim_{x\to \frac{\pi}{2}^+} \tan x,\ \lim_{x\to \frac{\pi}{2}^-} \tan x.$

3. Limits at infinity

(6) Evaluate

(a) $\lim_{x\to\infty} \frac{x^2+1}{x-3}$

(b) (Final, 2015) $\lim_{x\to-\infty} \frac{x+1}{x^2+2x-8}$

(c) (Quiz, 2015) $\lim_{x\to-\infty} \frac{3x}{\sqrt{4x^2+x}-2x}$