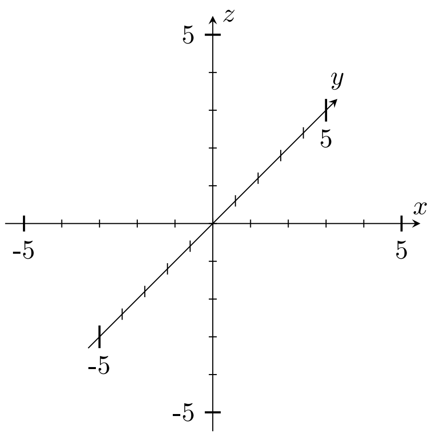
# Math 100C – WORKSHEET 10
# MULTIVARIABLE CALCULUS

## 1. Plotting in three dimensions

(1) ⋆ Plot the points $(2,1,3)$, $(-2,2,2)$ on the axes provided.




(2) Let $f(x,y) = e^{x^2+y^2}$.

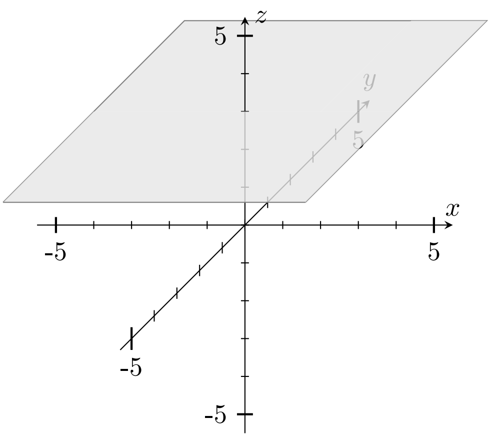
(a) ⋆ What are $f(0,-1)$? $f(1,2)$? Plot the point $(0,1,f(0,1))$ on the axes provided.

(b) ⋆ What is the *domain* of $f$ (that is: for what $(x,y)$ values does $f$ make sense?

(c) ⋆ What is the *range* of $f$ (that is: what values does it take)?

(3) ⋆⋆ What would the graph of $z = \sqrt{1-x^2-y^2}$ look like?

(4) ⋆ Which plane is this?




(A) $x = 3$
(B) $y = 3$
(C) $z = 3$
(D) none
(E) not sure

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*Date*: 24/11/2022, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

## 2. Partial derivatives

(5) (a) $\star$ Let $f(x) = 2x^2 - a^2 - 2$. What is $\frac{df}{dx}$?

(b) $\star$ Let $f(x) = 2x^2 - y^2 - 2$ where $y$ is a constant. What is $\frac{df}{dx}$?

(c) $\star$ Let $f(x, y) = 2x^2 - y^2 - 2$. What is the rate of change of $f$ as a function of $x$ *if we keep $y$ constant*?

(d) $\star$ What is $\frac{\partial f}{\partial y}$?

(6) Find the partial derivatives with respect to both $x, y$ of

(a) $\star$ $g(x, y) = 3y^2 \sin(x + 3)$

(b) $\star$ $h(x, y) = ye^{Axy} + B$

(7) The the gravitational *potential* due to a point mass $M$ (equivalently the electrical potential due to a point charge $M$) is given by the formula $U(x, y, z) = -\frac{GM}{r}$ where $r = \sqrt{x^2 + y^2 + z^2}$. Here $G$ is the universal gravitational constant (equivalently $G$ is the Coulomb constant).

(a) $\star$ The $x$-component of the field is given by the formula $F_x(x, y, z) = -\frac{\partial U}{\partial x}$. Find $F_x$

(b) $\star$ The magnitude of the field is given by $\left|\vec{F}\right| = \sqrt{F_x^2 + F_y^2 + F_z^2}$. How does it decay as a function of $r$?

(8) The *entropy* of an ideal gas of $N$ molecules at temperature $T$ and volume $V$ is

$$S(N,V,T) = Nk \log\left[\frac{VT^{1/(\gamma-1)}}{N\Phi}\right].$$

where $k$ is *Boltzmann's constant* and $\gamma, \Phi$ are constants that depend on the gas.

(a) $\star$ Find the *heat capacity at constant volume* $C_V = T\frac{\partial S}{\partial T}$.

(b) $\star\star\star$ Using the relation ("ideal gas law") $PV = NkT$ write $S$ as a function of $N, P, T$ instead. Differentiating with respect to $T$ **while keeping** $P$ **constant** determine the *heat capacity at constant pressure* $C_P = T\frac{\partial S}{\partial T}$.

Notations for the partial derivative include $\frac{\partial f}{\partial x}$, $\frac{\partial}{\partial x}f$, $\partial_x f$, $D_x f$,$f_x$.

(9) We can also compute second derivatives. For example $f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2}{\partial y \partial x}f$. Evaluate:

(a) $\star$ $h_{xx} = \frac{\partial^2 h}{\partial x^2} =$

(b) $\star$ $h_{xy} = \frac{\partial^2 h}{\partial y \partial x} =$

(c) $\star$ $h_{yx} = \frac{\partial^2 h}{\partial x \partial y} =$

(d) $\star$ $h_{yy} = \frac{\partial^2 h}{\partial y^2} =$

(10) $\star$ Repeat this exercise for the function $g$ from problem 2(a).

(11) You stand in the middle of a north-south street (say Health Sciences Mall). Let the $x$ axis run along the street (say oriented toward the south), and let the $y$ axis run across the street. Let $z = z(x, y)$ denote the height of the street surface above sea level.

(a) ⋆ What does $\frac{\partial z}{\partial y} = 0$ say about the street?

(b) ⋆ What does $\frac{\partial z}{\partial x} = 0.15$ say about the street?

(c) ⋆ You want to follow the street downhill. Which way should you go?

(d) The intersection of Health Sciences Mall and Agronomy Road is a local maximum. What does that say about $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ there?

## 3. Bonus (<u>**nonexaminable!**</u>): multivariable linear and higher approximation

**Definition 1.** A function $f(x, y)$ is *differentiable* at $x_0, y_0$ if we have a linear approximation $f(x, y) = f(x_0, y_0) + A(x - x_0) + B(y - y_0) + \text{small}$ as $(x, y) \to (x_0, y_0)$. We then have $A = \frac{\partial f}{\partial x}(x_0, y_0)$ and $B = \frac{\partial f}{\partial y}(x_0, y_0)$. The definition for functions of more than two variables is analogous.

(12) Let $f(x) = \sqrt{2 + x^2 + y^2}$.

(a) Write the linear approximation to $f$ about $(1, 1)$ and use that to estimate $f(1.1, 1.2)$.

(b) Write the linear approximation to $f$ about $(3, 5)$ and use that to estimate $f(2.8, 4.9)$.