1A3

3. The Derivative (20/9/2023)

Goals.

(1) The derivative at a point
(2) Tangent lines & linear approximations
(3) The derivative as a function

Last Time. <u>limits</u>

Write $\lim_{x\to a} f(x) = L$ if as x approaches a $(x \neq a)$ the values f(x) become closer to L.

<u>Extended sense:</u> $\lim_{x\to a} f(x) = \infty$ (or $-\infty$) ("vertical asymptote")

<u>Asymptotic behaviour:</u> $\lim_{x\to\infty} f(x) = L$ (or $-\infty$) ("horizontal asymptote")

<u>Continuity:</u> If $\lim_{x\to a} f(x) = f(a)$

(Can use this to <u>compute</u> limits if know f is continuous)

<u>promise:</u> formulas define continuous functions on their domain.

<u>Constant approximation:</u> $(1.2)^2 \approx 1^2$

$\sqrt{1.2} \approx \sqrt{1}$

at some level of accuracy.

Math 100A – WORKSHEET 3
THE DERIVATIVE

1. THREE VIEWS OF THE DERIVATIVE

(1) Let $f(x) = x^2$, and let $a = 2$. Then $(2, 4)$ is a point on the graph of $y = f(x)$.

(a) Let (x, x^2) be another point on the graph, close to $(2, 4)$. What is the slope of the line connecting the two? What is the limit of the slopes as $x \to 2$?

slope $= \frac{\Delta y}{\Delta x} = \frac{x^2-4}{x-2} = x+2 \xrightarrow[x\to 2]{} 4$

("zoom in to $y=x^2$ at $(2,4)$, see line of slope 4")

change variables

$\begin{cases} x = 2+h \\ h = x-2 \end{cases}$

(b) Let h be a small quantity. What is the asymptotic behaviour of $f(2 + h)$ as $h \to 0$? What about $f(2 + h) - f(2)$?

$f(2+h) = (2+h)^2 \sim 2^2 = f(2) = 4$

so look at $f(2+h) - f(2) = (2+h)^2 - 2^2 = 4 + 4h + h^2 - 4 = 4h + h^2 \sim 4h$

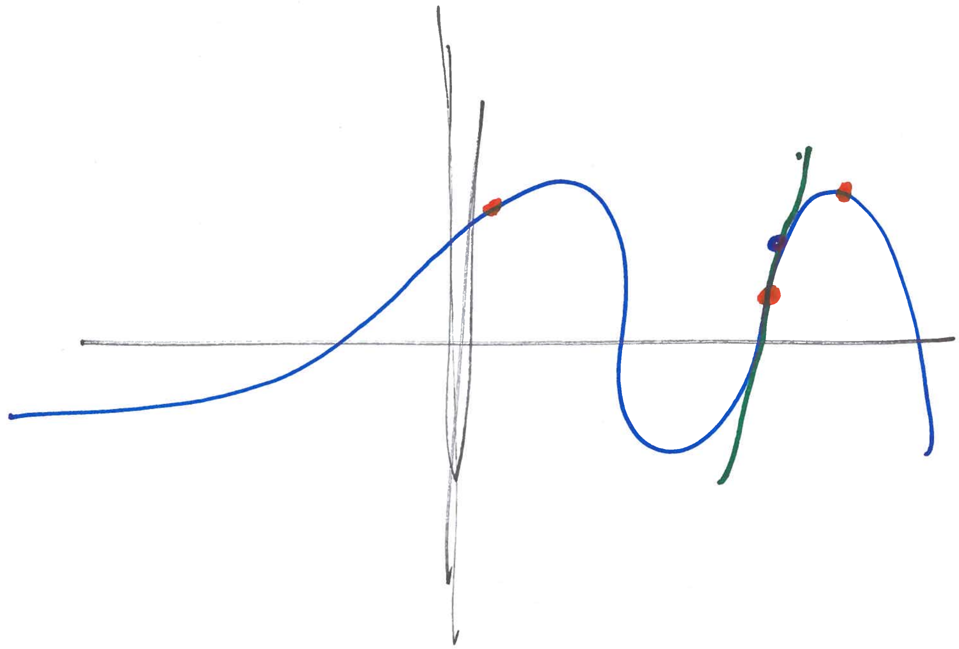
(slope 4)

(c) What is $\lim_{h\to 0} \frac{(2+h)^2-2^2}{h}$?

(d) What is the equation of the line tangent to the graph of $y = f(x)$ at $(2, 4)$?

Date: 20/9/2023, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

Example of linear approximation



W1 (a), (b)

Two pov:

① Geometry: at $x=2$, $y=x^2$ had slope 4

② Asymptotics: $f(2+h)-f(2) \sim 4h$

$\Leftrightarrow f(2+h) \approx f(2)+4h$

$\Leftrightarrow f(x) \approx 4+4(x-2)$

("if we wiggle x from 2 by h units
f(x) will wiggle from f(2) by about 4h units")

③ if $f(2+h)-f(2) \sim 4h$ then $\dfrac{f(2+h)-f(2)}{h} \xrightarrow[h\to 0]{} 4$

Definition: Let f be defined near $x=a$ (including at a). The **derivative** of f at a is the number

$$\left.\frac{df}{dx}\right|_{x=a} = f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

$$= \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

(if limit exists say f is **differentiable** at a)

Forward: Given formula for f, use limit to find $f'(a)$ (check if $f'(a)$ exists)

Backward: recognize a limit as a derivative.

Back to $y=x^2$, $a=2$. Found slope $=4$

1(d) line of slope 4 through $(2,4)$ is:

$$y = 4x-4$$

$$\Leftrightarrow \quad y-4 = 4(x-2)$$

$$\Leftrightarrow \quad y = \underset{f(2)}{4} + \underset{f'(2)}{4}(x-\underset{a=2}{2})$$

① tangent line

=

② linear approximation

(2) ⋆⋆ An enzymatic reaction occurs at rate $k(T) = T(40-T) + 10T$ where T is the temperature in degrees celsius. The current temperature of the solution is 20°C. Should we increase or decrease the temperature to increase the reaction rate?

Try $T = 20°C + h$

$$k(20) = 20(40-20) + 10 \cdot 20 = 600$$

$$k(20+h) = (20+h)(40-(20+h)) + 10(20+h)$$
$$= (20+h)(20-h) + 10 \cdot 20 + 10h$$
$$= 400 + 200 - h^2 + 10h$$
$$= 600 + 10h - h^2 \simeq 600 + 10h$$

~~sol ade~~ 10h > 0 if h > 0 so adjust upward

$\frac{dk}{dT} = 40 - T - T + 10$ so $k'(20) = 10 > 0$

so increase T to increase k.

if know diff rules

2. Definition of the derivative

Definition. $f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$ or $f(a+h) \approx f(a) + f'(a)h$

(3) Find $f'(a)$ if

(a) $\star$ $f(x) = x^2$, $a = 3$.

(b) $\star\star$ $f(x) = \frac{1}{x}$, any a.

Example:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{1}{h}\left(\frac{1}{a+h} - \frac{1}{a}\right)$$

$$= \lim_{h \to 0} \frac{1}{h}\left(\frac{a-(a+h)}{(a+h)a}\right) = \lim_{h \to 0} \frac{-h}{h \cdot a \cdot (a+h)} = \lim_{h \to 0} \frac{-1}{a(a+h)}$$

$$= -\frac{1}{a^2}$$

(c) ⋆⋆ $f(x) = x^3 - 2x$, any a (you may use $(a+h)^3 = a^3 + 3a^2h + 3ah^2 + h^3$).

(4) ⋆⋆ Express the limits as derivatives: $\lim_{h \to 0} \frac{\cos(5+h) - \cos 5}{h}$, $\lim_{x \to 0} \frac{\sin x}{x}$ $= \lim_{x \to 0} \frac{\sin x - \sin 0}{x - 0} = (\sin\theta)'(0)$

$\left(\frac{d}{d\theta} \cos\theta\right)(5)$

(5) ★★★ (Final, 2015, variant – gluing derivatives) Is the function

$$f(x) = \begin{cases} x^2 & x \leq 0 \\ x^2 \cos \frac{1}{x} & x > 0 \end{cases}$$

differentiable at $x = 0$?

Slope on left $\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{x^2}{x} = 0$

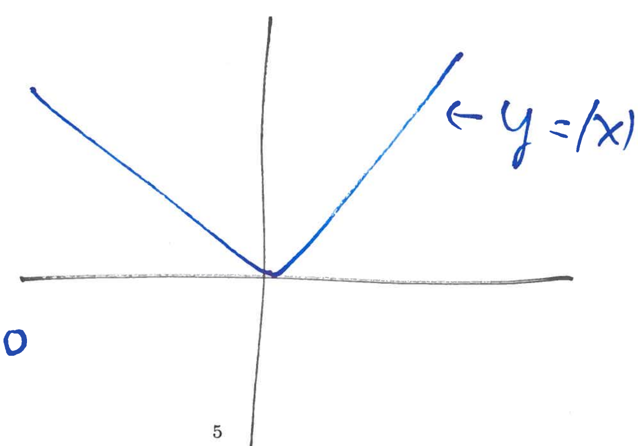
on right $\lim_{x \to 0^+} \frac{f(x)}{x} = \lim_{x \to 0^+} \frac{x^2 \cos(\frac{1}{x})}{x} = \lim_{x \to 0^+} x \cos(\frac{1}{x})$

$= 0$

since $x \to 0$ while $\cos(\frac{1}{x})$ is bounded

so $f'(0)$ exists and equals 0

Examples



not diff at $x = 0$

3. The tangent line

(6) ⋆⋆(Final, 2015) Find the equation of the line tangent to the function $f(x) = \sqrt{x}$ at $(4, 2)$.

(line through (4, 2) of slope $f'(4)$)

$f(x) = x^{\frac{1}{2}}$, so $f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$, so $f'(4) = \frac{1}{4}$.

so line is $y = 2 + \frac{1}{4}(x-4)$

(7) ⋆⋆(Final 2015) The line $y = 4x + 2$ is tangent at $x = 1$ to which function: $x^3 + 2x^2 + 3x$, $x^2 + 3x + 2$, $2\sqrt{x+3} + 2$, $x^3 + x^2 - x$, $x^3 + x + 2$, none of the above?

The line passes through (1, 6), has slope 4.

(8) ⋆ ⋆ ⋆ Find the lines of slope 3 tangent to the curve $y = x^3 + 4x^2 - 8x + 3$.

Let a be the point of tangency

Then $y'(a) = 3a^2 + 8a - 8 = 3$

now solve for a, find lines.

(9) ⋆ ⋆ ⋆ The line $y = 5x + B$ is tangent to the curve $y = x^3 + 2x$. What is B?

4. Linear approximation

$\boxed{\textbf{Definition. } f(a+h) \approx f(a) + f'(a)h}$

(10) Estimate

(a) $\star$ $\sqrt{1.2}$

let's extrapolate from $a=1$

$f(x) = \sqrt{x}$, $f(1) = 1$, $f'(1) = \frac{1}{2}$

$f'(x) = \frac{1}{2\sqrt{x}}$

so $f(x) \approx 1 + \frac{1}{2}(x-1)$ so $f(1.2) \approx 1 + \frac{1}{2} \cdot 0.2$

$f(1+h) \approx 1 + \frac{1}{2}h$, use $h = 0.2$. $= 1.1$

(b) $\star$ (Final, 2015) $\sqrt{8}$