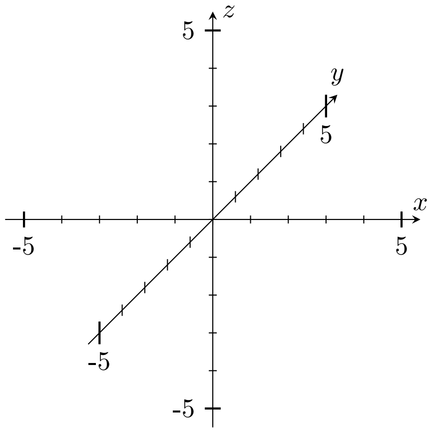
Math 100C – SOLUTIONS TO WORKSHEET 10
MULTIVARIABLE CALCULUS

1. Plotting in three dimensions

(1) Plot the points $(2, 1, 3)$, $(-2, 2, 2)$ on the axes provided.



(2) Let $f(x, y) = e^{x^2+y^2}$.

(a) What are $f(0, -1)$? $f(1, 2)$? Plot the point $(0, 1, f(0, 1))$ on the axes provided.

(b) What is the *domain* of f (that is: for what (x, y) values does f make sense?

Solution: f makes sense for all (x, y) – equivalenly that is on the plane $\mathbb{R}^2$.

(c) What is the *range* of f (that is: what values does it take)?

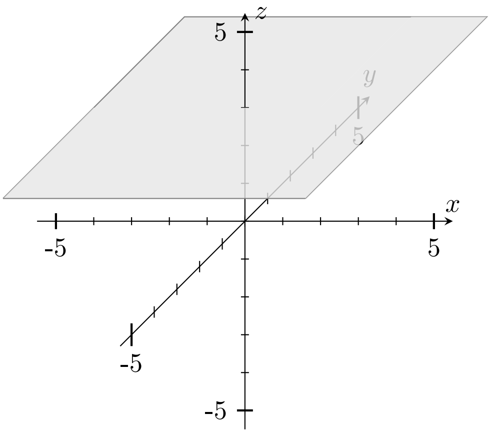
Solution: $x^2 + y^2$ takes all possible nonegative values, so $e^{x^2+y^2}$ takes all values in $[1, \infty)$.

(3) What would the graph of $z = \sqrt{1 - x^2 - y^2}$ look like?

Solution: This is the same as $x^2 + y^2 + z^2 = 1$ with $z \geq 0$, so the graph would be the upper half of the sphere of radius 1.

(4) Which plane is this?

Solution: (C) $z = 3$.



(A) $x = 3$
(B) $y = 3$
(C) $z = 3$
(D) none
(E) not sure

2. Partial derivatives

(5) (a) Let $f(x) = 2x^2 - a^2 - 2$. What is $\frac{df}{dx}$?

Solution: $\frac{df}{dx} = 4x$.

(b) Let $f(x) = 2x^2 - y^2 - 2$ where y is a constant. What is $\frac{df}{dx}$?

Solution: $\frac{df}{dx} = 4x$.

(c) Let $f(x, y) = 2x^2 - y^2 - 2$. What is the rate of change of f as a function of x *if we keep y constant*?

Solution: $\frac{\partial f}{\partial x} = 4x$.

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(d) What is $\frac{\partial f}{\partial y}$?
Solution: $\frac{\partial f}{\partial y} = -2y$.

(6) Find the partial derivatives with respect to both x, y of
(a) $g(x,y) = 3y^2\sin(x+3)$
Solution: $\frac{\partial g}{\partial x} = 3y^2\cos(x+3)$ (note that $3y^2$ is *constant* if y is) while $\frac{\partial g}{\partial y} = 6y\sin(x+3)$ (note that $\cos(x+3)$ is constant when x is constant).
(b) $h(x,y) = ye^{Axy} + B$
Solution: We have $\frac{\partial h}{\partial x} \overset{\text{linear}}{=} y\left(\frac{\partial}{\partial x}e^{Axy}\right) + \frac{\partial}{\partial x}B \overset{\text{chain}}{=} y \cdot Ay \cdot e^{Axy} = Ay^2e^{Axy}$
and $\frac{\partial h}{\partial y} \overset{\text{pdt}}{=} \left(\frac{\partial}{\partial y}y\right) \cdot e^{Axy} + y\left(\frac{\partial}{\partial y}e^{Axy}\right) = e^{Axy} + Axye^{Axy} = e^{Axy}(1+Axy)$.

(7) One model in labour economics has a production function $Q = \left[\alpha K^\delta + (1-\alpha)E^\delta\right]^{1/\delta}$. Here $\alpha, \delta > 0$ are parameters ($\alpha < 1$), K is the capital and E is the labour.
(a) Find the marginal product of capital: $\frac{\partial Q}{\partial K} =$
Solution: $\frac{\partial Q}{\partial K} = \frac{1}{\delta}\left[\alpha K^\delta + (1-\alpha)E^\delta\right]^{\frac{1}{\delta}-1}\alpha\delta K^{\delta-1} = \alpha\left[\alpha K^\delta + (1-\alpha)E^\delta\right]^{\frac{1}{\delta}-1}K^{\delta-1}$.
(b) Find the marginal product of labour: $\frac{\partial Q}{\partial E} =$
Solution: $\frac{\partial Q}{\partial E} = \frac{1}{\delta}\left[\alpha K^\delta + (1-\alpha)E^\delta\right]^{\frac{1}{\delta}-1}(1-\alpha)\delta E^{\delta-1} = (1-\alpha)\left[\alpha K^\delta + (1-\alpha)E^\delta\right]^{\frac{1}{\delta}-1}E^{\delta-1}$.

(8) We can also compute second derivatives. For example $f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2}{\partial y\partial x}f$. Evaluate:
(a) $h_{xx} = \frac{\partial^2 h}{\partial x^2} =$
Solution: We have $\frac{\partial^2 h}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial h}{\partial x}\right) = \frac{\partial}{\partial x}\left(Ay^2e^{Axy}\right) = A^2y^3e^{Axy}$.
(b) $h_{xy} = \frac{\partial^2 h}{\partial y\partial x} =$
Solution: We have $\frac{\partial^2 h}{\partial y\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial h}{\partial x}\right) = \frac{\partial}{\partial y}\left(Ay^2e^{Axy}\right) = 2Aye^{Axy} + A^2xy^2e^{Axy} = \left(2Ay + A^2xy^2\right)e^{Axy}$.
(c) $h_{yx} = \frac{\partial^2 h}{\partial x\partial y} =$
Solution: We have $\frac{\partial^2 h}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial h}{\partial y}\right) = \frac{\partial}{\partial x}\left(e^{Axy}(1+Axy)\right) = Aye^{Axy}(1+Axy) + e^{Axy} \cdot Ay = e^{Axy}\left(A^2xy^2 + 2Ay\right)$.
(d) $h_{yy} = \frac{\partial^2 h}{\partial y^2} =$
Solution: We have $\frac{\partial^2 h}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial h}{\partial y}\right) = \frac{\partial}{\partial y}\left(e^{Axy}(1+Axy)\right) = Axe^{xy}(1+Axy) + e^{Axy}(Ax) = Ax(2+Axy)e^{Ax}$.

(9) You stand in the middle of a north-south street (say Health Sciences Mall). Let the x axis run along the street (say oriented toward the south), and let the y axis run across the street. Let $z = z(x,y)$ denote the height of the street surface above sea level.
(a) What does $\frac{\partial z}{\partial y} = 0$ say about the street?
Solution: The street surface is level.
(b) What does $\frac{\partial z}{\partial x} = 0.15$ say about the street?
Solution: The street has a 15% grade sloping up toward the south: for each $1m$ we walk south we gain $0.15m$ in altitude.
(c) You want to follow the street downhill. Which way should you go?
Solution: Since altitude increases with increasing x (i.e. as you go south), you should go north.

3. Bonus (**nonexaminable!**): multivariable linear approximation

(10)