9. The Euler Scheme

Goals.

(1) Review ODE; *derive* an ODE
(2) Solving an ODE numerically

Last Time. Differential equations

Equation: (1) unknown function
(2) involves derivatives of the unknown

Basic examples: $y'=0$, $y'=1$, $y'=f(x)$

$y'=ry$ (solution: Ce^{rx}) ← memorize

(1) looked for explicit solutions using Ansätze
(2) involves working with unknown functions.

The Euler Scheme

① Have ODE $y'(x) = F(x; y(x))$ } inputs

② Have initial value $y(a) = y_0$

Goal: (Output) approximate $y(b)$ (working on $[a,b]$)

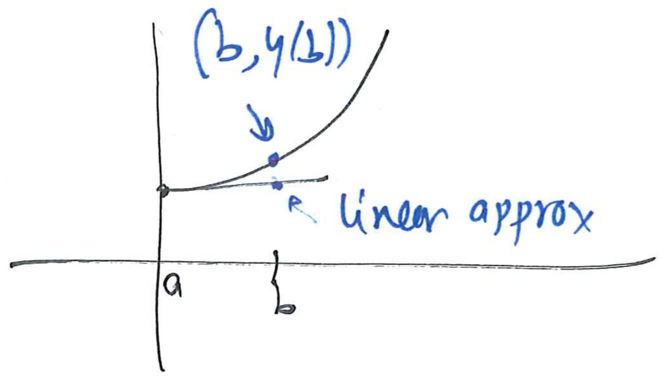
1-step method = linear approximation:

$$y(x) \approx y(a) + y'(a)(x-a)$$

$$= y_0 + F(a; y_0)(x-a)$$

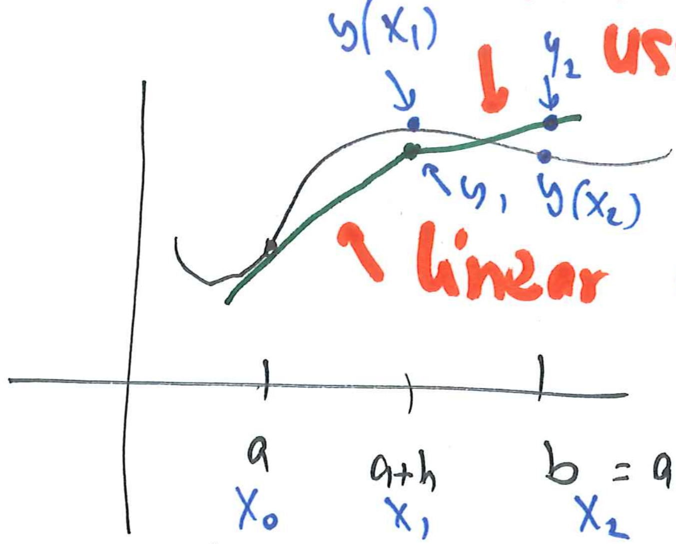
Example: say $y' = 3y$, $y(0) = 1$. Then $y'(0) = 3$

so $y(1) \approx 1 + 3(1-0) = 4$



2-step method

linear approx on [a+h, b] using y_1 not $y(a+h)$



linear approx on [a, a+h]

① Divide the interval into two halves of length $h = \frac{b-a}{2}$

② Two linear approximations:

$$y(a+h) \approx y(a) + y'(a)h = y_1$$

$$y(b) = y(a+2h) \approx \left[y(a+h) + y'(a+h)h \right]$$ ← true linear approx at $a+h = x_1$

$$y_2 \approx y_1 + F(a+h, y_1) \cdot h$$

Example: $y' = 3y$, $y(0) = 1 \Rightarrow y'(0) = 3$

so guess $y(\frac{1}{2}) \approx y_1 = 1 + 3 \cdot \frac{1}{2} = 2\frac{1}{2}$

$\Rightarrow$ guess $y'(\frac{1}{2}) \approx 3 \cdot 2\frac{1}{2} = 7\frac{1}{2}$

step size

guess $y(1)$:

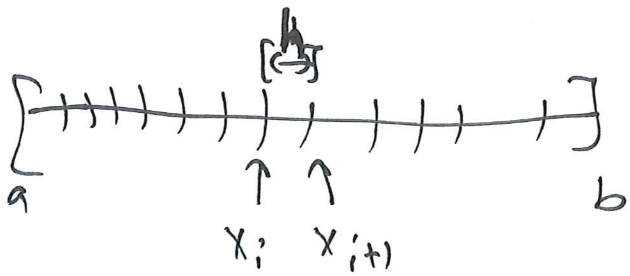
$$y(1) \approx y_2 = 2\tfrac{1}{2} + 7\tfrac{1}{2} \cdot \tfrac{1}{2} = 6\tfrac{1}{4}$$

n-step method

Have $\begin{cases} \text{ODE} & y' = F(x, y) \\ \text{initial value} & y_0 \\ \text{interval} & [a, b]. \end{cases}$

① Divide the interval into n steps of length $h = \frac{b-a}{n}$

$x_0 = a, \; x_1 = a+h, \; x_2 = a+2h, \ldots, x_i = a+ih, \ldots, x_n = b$



② Start with y_0, make

$$y_1 = y_0 + F(x_0; y_0) \cdot h$$

$$y_2 = y_1 + F(x_1; y_1) \cdot h$$

$$\vdots$$

$$y_{i+1} = y_i + F(x_i; y_i) h$$

$$\vdots$$

$$y_n = y_{n-1} + F(x_{n-1}; y_{n-1}) \cdot h$$

Example: continuously compounded interest

(Bernoulli 1683)

WS ①, ②

Conclusion: If compounding period h is very short then the balance $y(t)$ looks to have linear approx with slope $30\%\ y(t)$

$\Rightarrow$ in the limit $h \to 0$

$$y'(t) = 30\%\ y(t)$$

(solution: $y(t) = y_0 \cdot e^{0.3t}$)

POV 1: $y(1) \approx y(0) \cdot \left(1 + \frac{r}{n}\right)^n$

interest rate if compound n times

$n \to \infty$

$$e^r = \lim_{n \to \infty} \left(1 + \frac{r}{n}\right)^n$$

(Aside.)

1. Compound interest (Bernoulli 1683)

(1) Suppose you have a \$100 bank balance which earns an annual interest rate of 30%.

(a) Suppose the interest is paid once, at the end of the year. How much would your balance be at that time? $\$130 = \$100 \cdot (1+30\%)$

$= \$100 + 30\% \cdot \$100.$

two points of view

(b) Suppose instead that interest is paid four times a year. What is the quarterly interest *rate*? What would the balance be at the end of the first quarter? rate is $\frac{1}{4} \cdot 30\% = 7.5\%$.

balance: $107.5 = 100 \cdot (1 + .075)$

$= 100 + 0.075 \cdot 100.$

Date: 17/11/2022, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

(c) Suppose further that interest is *compounded*: after every quarter the interest is added to the balance. What would be the balance at the end of the year?

POV 1: multiply by $(1+0.075)$ each quarter, end with $100 \cdot (1{,}075)^4$.

POV 2: $y_{i+1} = y_i + 0.075 \cdot y_i$

y_{i+1}: balance at i+1'st quarter

y_i: balance at ith quarter

(d) Suppose instead that interest is compounded *daily* and that at a particular day the balance is y dollars. What is the balance the next day?

$$y_{i+1} = y_i \left(1 + \frac{30\%}{365}\right)$$

← daily interest rate

$$= y_i + 30\% \cdot y_i \frac{1}{365}$$

(2) Suppose interest is compounded *continuously* and that at a particular time y the balance is $y(t)$ dollars, where t is measured in years.

(a) What is the approximate interest rate for the period between times $t, t+h$ if h is very small?

Interest rate for period h is $\frac{h}{1} \cdot 30\%$

whole year

(b) What is the balance at time $t+h$?

$$y(t+h) \approx y(t) + (30\% \cdot h) \cdot y(t)$$

$$\approx y(t) + 30\% \cdot y(t) \cdot h$$

2. Further examples

From now on let the interest rate by r.

(3) Suppose that in addition to the interest we also have a constant income stream of b dollars per ~~month~~. year

(a) What differential equation expresses our bank balance now?

interest income

$$\underbrace{y' = ry + b}$$

fixed stream

(b) What is the general solution (hint: use an ansatz of the form $Ce^{rt} + B$). What is the solution that has $y(0) = y_0$?

(4) Suppose instead that our income stream is seasonal, so that the differential equation is $y' = ry + b\sin(2\pi t)$. Find the general solution and the solution satisfying $y(0) = y_0$ using an Ansatz of the form $Ae^{rt} + B\sin(2\pi t) + C\cos(2\pi t)$.