4. Computing derivatives (6/10/2022)

Goals.

(1) Combining linear approximations
(2) The product and quotient rules

Last Time.

If f cts at x can ask how f approaches $f(x)$ near x, i.e. look at $f(x+h)-f(x)$. Sometimes this has linear asymptotics: $f(x+h)-f(x) \sim ah$ (as $h \to 0$, x fixed)

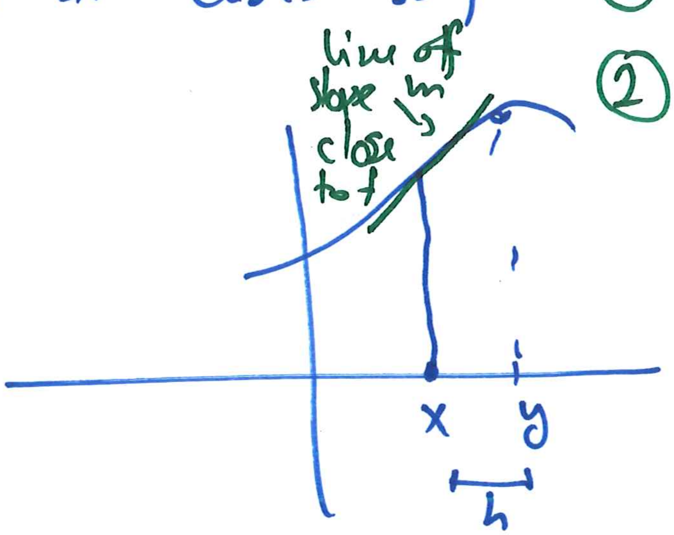
$\iff f(x+h) \approx f(x) + mh$

$\iff f(y) \approx f(x) + m(y-x) \iff f(x) \approx f(a) + m(x-a)$

$\iff \lim_{y\to x} \frac{f(y)-f(x)}{y-x} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = m$

in all cases say: (1) f is differentiable at x

(2) the derivative $f'(x) = m$



Math 100C – WORKSHEET 3
ARITHMETIC OF THE DERIVATIVE

1. Review of the derivative

(1) Expand $f(x+h)$ to linear order in h for the following functions and read the derivative off:

(a) $f(x) = bx$

$f(x+h) = b(x+h) = bx + bh$

linear in h, slope b

(b) $g(x) = ax^2$

$g(x+h) = a(x+h)^2 = ax^2 + 2axh + ah^2 \approx ax^2 + (2ax) \cdot h$

to 1'st order in h as $h \to 0$

slope : $2ax$

(c) $h(x) = ax^2 + bx$.

method 1: $h(x+h) = a(x+h)^2 + b(x+h) = ax^2 + 2axh + ah^2 + bx + bh$

$= (ax^2 + bx) + (2ax+b)h + ah^2 \approx (ax^2+bx) + (2ax+b)h$

to 1st order as $h \to 0$

method 2:

$f(x+h) \approx f(x) + bh$

$+\ g(x+h) \approx g(x) + (2ax)h$

$\boxed{\text{add}}$

$h(x+h) \approx (f(x) + g(x)) + (2ax+b)h$ so $h'(x) = f'(x) + g'(x)$

WS 1 (a), (b), (c)

Takeaway: since can get a linear approx to $f+g$ by approximating f, g and adding, and since slope (linear + linear) is the sum of the slopes, get

$$\boxed{(f+g)'(x) = f'(x) + g'(x)}$$

(more generally, if $\alpha, \beta \in \mathbb{R}$ then

$$\boxed{(\alpha f + \beta g)'(x) = \alpha f'(x) + \beta g'(x)}$$

("linearity of $\frac{d}{dx}$")

(d) $i(x) = \frac{1}{b+x}$

$$i(x+h) = \frac{1}{b+x+h} - \frac{1}{b+x} + \frac{1}{b+x} = \frac{1}{b+x} + \frac{(b+x)-(b+x+h)}{(b+x+h)(b+x)}$$

$$= \frac{1}{b+x} - \frac{h}{(b+x+h)(b+x)} \approx \frac{1}{b+x} - \frac{h}{(b+x)^2}$$

$$\frac{1}{(b+x+h)(b+x)} \sim \frac{1}{(b+x)^2}$$

as $h \to 0$.

so slope is $-\frac{1}{(b+x)^2}$.

(e) $j(x) = 4x^4 + 5x$ (hint: use the known linear approximation to $2x^2$)

approx of 1(b)

$$j(x) = (2x^2)^2 + 5x$$

$$j(x+h) = (2(x+h)^2)^2 + 5(x+h) \approx (2x^2 + 4xh)^2 + 5(x+h)$$

square again . . .

Product & Quotient rules

Saw can combine approximations additively:

if $f(x+h) \approx f(x) + f'(x)h$
$g(x+h) \approx g(x) + g'(x)h$

$\Downarrow$

$$(\alpha f+\beta g)(x+h) \approx (\alpha f+\beta g)(x) + (\alpha f'(x)+\beta g'(x))h$$

$$\Rightarrow (\alpha f+\beta g)'(x) = \alpha f'(x) + \beta g'(x).$$

What about $(f\cdot g)(x+h)$?

$$(f\cdot g)(x+h) = f(x+h)\cdot g(x+h) \approx (f(x)+f'(x)h)(g(x)+g'(x)h)$$

$$\approx f(x)g(x) + f(x)g'(x)h + f'(x)h\,g(x) + f'(x)g'(x)h^2$$

$$\approx (f\cdot g)(x) + (f(x)g'(x) + f'(x)g(x))h$$

to 1st order as $h \to 0$

so $\boxed{(f\cdot g)'(x) = f(x)g'(x) + f'(x)g(x)}$

"product rule"

(side idea: to approximate $A\cdot B$ can approx A, approx B, then multiply approximations)

Similarly:

$$\frac{f}{g}(x+h) = \frac{f(x+h)}{g(x+h)} = \frac{f(x)+f'(x)h}{g(x)+g'(x)h} \approx \cdots$$

$$\cdots \approx \frac{f(x)}{g(x)} + \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)} \cdot h$$

to 1st order in h
as $h \to 0$

$\Rightarrow$ if $f'(x)$, $g'(x)$ exist & $g(x) \neq 0$ then $\left(\frac{f}{g}\right)'(x)$ exists and

$$\boxed{\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}}$$

"quotient rule"

Point of calculations:

① To show that the "rules" really are laws of nature

② To illustrate idea of combining approximations.

(no proofs in MATH 100C)

2. Arithmetic of derivatives

(2) Differentiate

(a) $f(x) = 6x^{\pi} + 2x^{e} - x^{7/2}$

by linearity

$$\frac{df}{dx} = 6\frac{d(x^{\pi})}{dx} + 2\frac{d(x^{e})}{dx} - \frac{d}{dx}x^{7/2} = 6\cdot\pi\cdot x^{\pi-1} + 2e\cdot x^{e-1} - \frac{7}{2}x^{5/2}$$

(b) (Final, 2016) $g(x) = x^2e^x$ (and then also x^ae^x)

this is a product so

$$\frac{dg}{dx} = \frac{d(x^2)}{dx}\cdot e^x + x^2\cdot\frac{d(e^x)}{dx} = 2xe^x + x^2e^x$$

$$= x(2+x)e^x$$

sometimes convenient to factor f'

(e.g. helps in solving $f'(x)=0$, $f'(x)>0$, $f'(x)<0$)

(c) (Final, 2016) $h(x) = \frac{x^2+3}{2x-1}$

$$h'(x) = \frac{2x\cdot(2x-1)-(x^2+3)\cdot 2}{(2x-1)^2} = \frac{4x^2-2x-2x^2-6}{(2x-1)^2} =$$

$$= \frac{2x^2-2x-6}{(2x-1)^2} = 2\frac{x^2-x-3}{(2x-1)^2}$$

(d) $\frac{x^2+A}{\sqrt{x}}$

$$\frac{x^2+A}{\sqrt{x}} = \frac{x^2}{\sqrt{x}} + \frac{A}{\sqrt{x}} = x^{3/2} + Ax^{-\frac{1}{2}}$$

So $\left(\frac{x^2+A}{\sqrt{x}}\right)' = \frac{3}{2}x^{1/2} - \frac{1}{2}Ax^{-3/2}$

(3) Let $f(x) = \frac{x}{\sqrt{x}+A}$. Given that $f'(4) = \frac{3}{16}$, give a quadratic equation for A.

We have $f'(x) = \frac{1\cdot(\sqrt{x}+A) - x\cdot\frac{1}{2\sqrt{x}}}{(\sqrt{x}+A)^2} = \frac{\sqrt{x}+A-\frac{1}{2}\sqrt{x}}{(\sqrt{x}+A)^2} =$

$$= \frac{\frac{1}{2}\sqrt{x}+A}{(\sqrt{x}+A)^2}$$

So $\frac{3}{16} = f'(4) = \frac{\frac{1}{2}\sqrt{4}+A}{(\sqrt{4}+A)^2} = \frac{1+A}{(2+A)^2}$

i.e. $\frac{3}{16} = \frac{1+A}{(2+A)^2} \Rightarrow \frac{3}{16}(2+A)^2 = 1+A$ . . .

(4) Suppose that $f(1) = 1$, $g(1) = 2$, $f'(1) = 3$, $g'(1) = 4$.

(a) What are the linear approximations to f and g at $x = 1$? Use them to find the linear approximation to fg at $x = 1$.

We have (b) Find $(fg)'(1)$ and $\left(\frac{f}{g}\right)'(1)$.

$$(fg)'(1) \underset{\text{pdt rule}}{=} f'(1)g(1) + f(1)\cdot g'(1) = 3\cdot 2 + 1\cdot 4 = 10$$

$$\left(\frac{f}{g}\right)'(1) = \frac{f'(1)g(1) - f(1)g'(1)}{(g(1))^2} = \frac{3\cdot 2 - 1\cdot 4}{2^2} = \frac{1}{2}$$

<u>Takeaway</u>: just do what the question asks

(5) Evaluate

(a) $(x \cdot x)'$ and $(x') \cdot (x')$. What did we learn?

$(x \cdot x)' = (x^2)' = 2x$; $(x') \cdot (x') = 1 \cdot 1 = 1$

$(f \cdot g)' \neq f' \cdot g'$

(b) $\left(\frac{x}{x}\right)'$ and $\frac{(x')}{(x')}$. What did we learn?

$(1)' = 0$ $\qquad$ $\frac{1}{1} = 1$

$\left(\frac{f}{g}\right)' \neq \frac{f'}{g'}$

(6) The *Lennart–Jones potential* $V(r) = \epsilon\left(\left(\frac{R}{r}\right)^{12} - 2\left(\frac{R}{r}\right)^{6}\right)$ models the electrostatic potential energy of a diatomic molecule. Here $r > 0$ is the distance between the atoms and $\epsilon, R > 0$ are constants.

(a) What are the asymptotics of $V(r)$ as $r \to 0$ and as $r \to \infty$?

As $r \to 0$, $V(r) \sim \varepsilon\left(\frac{R}{r}\right)^{12}$

As $r \to \infty$ $V(r) \sim -2\varepsilon\left(\frac{R}{r}\right)^{6}$

(b) Sketch a plot of $V(r)$.