Math 100 – SOLUTIONS TO WORKSHEET 17
THE MEAN VALUE THEOREM; LINEAR APPROXIMATION

1. Average slope vs Instantenous slope

(1) Let $f(x) = e^x$ on the interval $[0, 1]$. Find all values of c so that $f'(c) = \frac{f(1)-f(0)}{1-0}$.

Solution: $\frac{f(1)-f(0)}{1-0} = \frac{e-1}{1} = e - 1$ and $f'(x) = e^x$ so if $e^c = e - 1$ we have $c = \log(e - 1)$ and indeed $1 < e - 1 < e$ means $0 < \log(e - 1) < 1$.

(2) Let $f(x) = |x|$ on the interval $[-1, 2]$. Find all values of c so that $f'(c) = \frac{f(2)-f(-1)}{2-(-1)}$

Solution: There is no such value: $\frac{f(2)-f(-1)}{2-(-1)} = \frac{2-1}{3} = \frac{1}{3}$ but $f'(x)$ only takes the values ± 1.

2. The Mean Value Theorem

(3) Show that $f(x) = 3x^3 + 2x - 1 + \sin x$ has exactly one real zero. (Hint: let a, b be zeroes of f. The MVT will find c such that $f'(c) =?$)

Solution: We first check there is at least one zero. For this note that f is continuous (it's defined by formula), and that $f(10) = 3009 + \sin 10 \geq 3008 > 0$ and $f(-10) = -3021 - \sin 10 \leq -3020 < 0$. By the IVT f has a zero a between $(-10, 10)$. Now suppose there were at least two zeros; calling two of them a, b we'd have $f(a) = f(b) = 0$. The function f is everywhere differentiable (defined by formula), so by the MVT there is c between a, b such that $f'(c) = \frac{f(b)-f(a)}{b-a} = 0$. But $f'(x) = 9x^2+2+\cos x > 0$ for all x.

(4) (Final, 2015)

(a) Suppose f, f', f'' are all continuous. Suppose f has at least three zeroes. How many zeroes must f', f'' have?

Solution: Suppose $f(a) = f(b) = 0$. Since f is everywhere differentiable, by the MVT there is x between a, b such that $f'(x) = \frac{f(b)-f(a)}{b-a} = 0$. Now if $a < b < c$ are zeroes of f we find a zero of f' between (a, b) and between (b, c) (so f' has at least two zeroes) and then f'' has a zero between the two zeroes of f', so f'' has at least one zero.

(b) [Show that $2x^2 - 3 + \sin x + \cos x = 0$ has at least two solutions]

Solution: See IVT worksheet

(c) Show that the equation has at most two solutions.

Solution: Suppose $f(x) = 2x^2 - 3 + \sin x + \cos x$ had three zeroes. Then by part (a), $f''(x)$ would have a zero. But $f''(x) = 4 - \sin x - \cos x \geq 4 - 1 - 1 = 2 > 0$ is nowhere vanishing.

(5) (Final, 2012) Suppose $f(1) = 3$ and $-3 \leq f'(x) \leq 2$ for $x \in [1, 4]$. What can you say about $f(4)$?

Solution: Since f is everywhere differentiable, by the MVT there is $c \in (1, 4)$ such that

$$\frac{f(4) - f(1)}{4 - 1} = f'(c)\,.$$

It follows that

$$-3 \leq \frac{f(4) - f(1)}{3} \leq 2$$

and hence

$$-6 \leq f(1) + (-3) \cdot 3 \leq f(4) \leq f(1) + 2 \cdot 3 = 9\,.$$

(6) Show that $|\sin a - \sin b| \leq |a - b|$ for all a, b.

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Solution: The claim is automatic if $a = b$ so assume $a \neq b$. Since $f(x) = \sin x$ is everywhere differentiable, for any $a \neq b$ we may apply the MVT to find c between them such that $\frac{\sin a - \sin b}{a-b} = f'(c) = \cos c$. It follows that

$$\frac{|\sin a - \sin b|}{|a-b|} = |\cos c| \leq 1$$

and the claim follows.

(7) Let $x > 0$. Show that $e^x > 1 + x$ and that $\log(1+x) < x$.

Solution: The function e^x is everywhere differentiable and its derivative is e^x. For $x > 0$ we therefore have $0 < c < x$ such that

$$\frac{e^x - e^0}{x - 0} = e^c > 1\,.$$

(the latter since $c > 0$). It follows that $e^x > x + e^0 = x + 1$.
Similarly, the function $\log(y)$ is differentiable on $[1, \infty)$ with derivative $\frac{1}{y}$. It follows that for $x > 0$ we have d in the interval $1 < d < 1 + x$ such that

$$\frac{\log(1+x) - \log 1}{(1+x) - 1} = \frac{1}{d} < 1$$

(the latter since $d > 1$). Since $\log 1 = 0$ and $(1+x) - 1 = x$ it follows that

$$\log(1+x) < x\,.$$

3. The Linear Approximation

(8) Use a linear approximation to estimate

(a) $\sqrt{1.2}$
Solution: Let $f(x) = \sqrt{x}$ so that $f'(x) = \frac{1}{2\sqrt{x}}$. Then $f(1) = 1$ and $f'(1) = \frac{1}{2}$ so $f(1.2) \approx f(1) + f'(1) \cdot 0.2 = 1 + \frac{1}{2} \cdot 0.2 = 1.1$.
Better: $f(1.21) = 1.1$ and $f'(1.21) = \frac{1}{2.2}$ so $f(1.2) = f(1.21 - 0.01) \approx 1.1 - 0.01 \cdot \frac{1}{2.2} \approx 1.09545$.

(b) (Final, 2015) $\sqrt{8}$
Solution: Using the same f we have $f(9-1) \approx f(9) + f'(9) \cdot (-1) = 3 - \frac{1}{6} = 2\frac{5}{6}$.

(c) (Final, 2016) $(26)^{1/3}$
Solution: Let $f(x) = x^{1/3}$ so that $f'(x) = \frac{1}{3}x^{-2/3}$. Then $f(27) = 3$ and $f'(27) = \frac{1}{3 \cdot 27^{2/3}} = \frac{1}{27}$ so

$$f(26) = f(27-1) \approx f(27) + (-1) \cdot f'(27) = 3 - \frac{1}{27} = 2\frac{26}{27}\,.$$

(d) $\log 1.07$
Solution: Let $f(x) = \log x$ so that $f'(x) = \frac{1}{x}$. Then $f(1) = 0$ and $f'(1) = 1$ so $f'(1.1) \approx 0.07$.