Math 100 – SOLUTIONS TO WORKSHEET 10
LOGARITHMIC AND IMPLICIT DIFFERENTIATION

1. Review of Logarithms

(1) $\log\left(e^{10}\right) =$ $\qquad\qquad$ $\log(2^{100}) =$

Solution: $\log e^{10} = 10$ while $\log(2^{100}) = 100\log 2$.

(2) A variant on *Moore's Law* states that computing power doubles every 18 months. Suppose computers today can do N_0 operations per second.

(a) Write a formula predicting the future:

- Computers t years from now will be able to do $N(t)$ operations per second where

$$N(t) =$$

Solution: Since there is a doubling every 18 months, there will be $t/1.5$ doublings in t years and $N(t) = N_0 2^{t/1.5}$.

(b) A computing task would take 10 years for today's computers. Suppose we wait 3 years and then start the computation. When will we have the answer?

Solution: There will be two doublings in 3 years, so we will have the answer $3 + \frac{10}{2^2} = 3 + \frac{10}{4} =$ 5.5 years from now.

(c) At what time will computers be powerful enough to complete the task in 6 months?

Solution: The computers of t years from now will be able to complete the task in $10 \cdot 2^{-t/1.5}$ years, so we need to find t such that

$$10 \cdot 2^{-t/1.5} = \frac{1}{2}\,.$$

This is equivalent to $2^{t/1.5} = 20$, and taking logarithms gives

$$\frac{t}{1.5}\log 2 = \log 20$$

and hence

$$\boxed{t = 1.5\frac{\log 20}{\log 2}}\,.$$

Solution: Can also write $2^{-t/1.5} = \frac{1}{20}$ and take logarithms to get $-\frac{t}{1.5} = \frac{\log\frac{1}{20}}{\log 2}$ so that $t = -1.5\frac{\log\frac{1}{20}}{\log 2} = 1.5\frac{\log 20}{\log 2}$ since $\log\frac{1}{20} = -\log 20$.

2. Differentiation

$$\boxed{(\log x)' = \frac{1}{x}}$$

(1) Differentiate

(a) $\frac{\mathrm{d}(\log(ax))}{\mathrm{d}x} =$ $\qquad\qquad$ $\frac{\mathrm{d}}{\mathrm{d}t}\log\left(t^2 + 3t\right) =$

Solution: By the chain rule, the derivatives are: $\frac{1}{ax} \cdot a = \frac{1}{x}$ and $\frac{1}{t^2+3t} \cdot (2t+3) = \frac{2t+t}{t^2+3t}$. We can also use the logarithm laws first: $\log(ax) = \log a + \log x$ so $\frac{\mathrm{d}}{\mathrm{d}x}(\log ax) = \frac{\mathrm{d}}{\mathrm{d}x}(\log a) + \frac{\mathrm{d}}{\mathrm{d}x}(\log x) = \frac{1}{x}$ since $\log a$ is constant if a is. Similarly, $\log(t^2+3t) = \log t + \log(t+3)$ so its derivative is $\frac{1}{t} + \frac{1}{t+3}$.

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(b) $\frac{\mathrm{d}}{\mathrm{d}x}x^2\log(1+x^2) =$ $\frac{\mathrm{d}}{\mathrm{d}r}\frac{1}{\log(2+\sin r)} =$

Solution: Applying the product rule and then the chain rule we get: $\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\log(1+x^2)\right) = 2x\log(1+x^2) + x^2\frac{1}{1+x^2}\cdot 2x = 2x\log(1+x^2) + \frac{2x^3}{1+x^2}$. Using the quotient rule and the chain rule we get

$$\frac{\mathrm{d}}{\mathrm{d}r}\frac{1}{\log(2+\sin r)} = -\frac{1}{\log^2(2+\sin r)}\cdot\frac{1}{2+\sin r}\cdot\cos r = -\frac{\cos r}{(2+\sin r)\log^2(2+\sin r)}\,.$$

(2) (Logarithmic differentiation) differentiate $y = \left(x^2+1\right)\cdot\sin x\cdot\frac{1}{\sqrt{x^3+3}}\cdot e^{\cos x}$.

Solution: We have

$$\begin{aligned}\log y &= \log\left(x^2+1\right) + \log(\sin x) + \log\left(\frac{1}{\sqrt{x^2+3}}\right) + \log\left(e^{\cos x}\right)\\ &= \log\left(x^2+1\right) + \log\left(\sin x\right) - \frac{1}{2}\log\left(x^2+3\right) + \cos x\,.\end{aligned}$$

Differentiating with respect to x gives:

$$\frac{y'}{y} = \frac{2x}{x^2+1} + \frac{\cos x}{\sin x} - \frac{1}{2}\frac{2x}{x^2+3} - \sin x$$

and solving for y' finally gives

$$y' = \left(\frac{2x}{x^2+1} + \frac{\cos x}{\sin x} - \frac{x}{x^2+3} - \sin x\right)\cdot\left(x^2+1\right)\cdot\sin x\cdot\frac{1}{\sqrt{x^3+3}}\cdot e^{\cos x}\,.$$

(3) Differentiate using $\boxed{f' = f\times(\log f)'}$

(a) x^x

Solution: If $y = x^x$ then $\log y = x\log x$. Differentiating with respect to x gives $\frac{1}{y}y' = \log x + x\cdot\frac{1}{x} = \log x + 1$ so $y' = y\left(\log x+1\right) = x^x\left(\log x+1\right)$.

Solution: By the rule, $\frac{\mathrm{d}}{\mathrm{d}x}\left(x^x\right) = x^x\frac{\mathrm{d}}{\mathrm{d}x}\left(\log(x^x)\right) = x^x\left(\log x+1\right)$.

Solution: We have $x^x = \left(e^{\log x}\right)^x = e^{x\log x}$. Applying the chain rule we now get $\left(x^x\right)' = e^{x\log x}\left(\log x+1\right) = x^x\left(\log x+1\right)$.

(b) $(\log x)^{\cos x}$

Solution: By the logarithmic differentiation rule we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\log x)^{\cos x} &= (\log x)^{\cos x}\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left(\cos x\log(\log x)\right)\\ &= -\sin x\log\log x\,(\log x)^{\cos x} + (\log x)^{\cos x}\cos x\frac{1}{\log x}\frac{1}{x}\\ &= -\sin x\log\log x\,(\log x)^{\cos x} + \cos x\,(\log x)^{\cos x-1}\frac{1}{x}\,.\end{aligned}$$

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(c) (Final, 2014) Let $y = x^{\log x}$. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x only.

Solution: By the logarithmic differentiation rule we have

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= y\frac{\mathrm{d}\log y}{\mathrm{d}x} = x^{\log x}\frac{\mathrm{d}}{\mathrm{d}x}\left(\log x\cdot\log x\right)\\ &= x^x\left(2\log x\cdot\frac{1}{x}\right) = 2\log x\cdot x^{x-1}\,.\end{aligned}$$

3. Implicit Differentiation

(1) Find the line tangent to the curve $y^2 = 4x^3 + 2x$ at the point $(2,6)$.

Solution: Differentiating with respect to x we find $2y\frac{\mathrm{d}y}{\mathrm{d}x} = 12x^2 + 2$, so that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{6x^2+1}{y}$. In particular at the point $(2,6)$ the slope is $\frac{25}{6}$ and the line is

$$y = \frac{25}{6}(x-2) + 6\,.$$

(2) (Final, 2015) Let $xy^2 + x^2y = 2$. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ at the point $(1, 1)$.

Solution: Differentiating with respect to x we find $y^2 + 2xy\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy + x^2\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ along the curve. Setting $x = y = 1$ we find that, at the indicated point,

$$3 + 3\frac{\mathrm{d}y}{\mathrm{d}x}\Big|_{(1,1)} = 0$$

so

$$\frac{\mathrm{d}y}{\mathrm{d}x}\Big|_{(1,1)} = -1\,.$$

(3) (Final 2012) Find the slope of the line tangent to the curve $y + x\cos y = \cos x$ at the point $(0, 1)$.

Solution: Differentiating with respect to x we find $y' + \cos y - x\sin y \cdot y' = -\sin x$, so that $y' = -\frac{\sin x+\cos y}{1-x\sin y} = \frac{\sin x+\cos y}{x\sin y-1}$. Setting $x = 0$, $y = 1$ we get that at that point $y' = \frac{\cos 1}{-1} = -\cos 1$.

(4) Find y'' (in terms of x, y) along the curve $x^5 + y^5 = 10$ (ignore points where $y = 0$).

Solution: Differentiating with respect to x we find $5x^4 + 5y^4y' = 0$, so that $y' = -\frac{x^4}{y^4}$. Differentiating again we find

$$y'' = -\frac{4x^3}{y^4} + \frac{4x^4y'}{y^5} = -\frac{4x^3}{y^4} - \frac{4x^8}{y^9}\,.$$

(5) Find y' if $(x + y)\sin(xy) = x^2$.

Solution: Differentiating with respect to x we find $(1 + y')\sin(xy)+(x+y)\cos(xy)(y+xy') = 2x$, so that

$$y'\left[\sin(xy) + x(x + y)\cos(xy)\right] = 2x - \left[\sin(xy) + y(x + y)\cos(xy)\right]$$

so that

$$y' = \frac{2x - \sin(xy) - y(x + y)\cos(xy)}{\sin(xy) + x(x + y)\cos(xy)}\,.$$