Math 100, lecture 24, 7/12/2021

Review 3

Q: Remainder estimates

Say we have $T_1(x) = f(a) + f'(a)(x-a)$ (linear approx)

the quadratic approx is

approximate $f(x) \approx \underbrace{T_1(x) + \frac{1}{2!} f''(a)(x-a)^2}_{T_2(x)}$

The remainder thm: $f(x) = T_1(x) + \frac{1}{2!} f''(c)(x-a)^2$, for some c between a, x

equal

Remainder tell us ~~[illegible]~~ how far $T_1(x)$ is from $f(x)$

If we want to know how far f(x) is from the approximation $T_1(x)$, we estimate $R_1(x) = \frac{1}{2!} f''(c)(x-a)^2$

NOT THE SAME AS TAKING LARGER OF $f''(a)$, $f''(x)$

We don't know what c is. We ~~don't~~ can't calculate $f''(c)$ exactly, we can say thing like "for any t between a, x, $f''(t) < M$"

Example: $f^{(3)}(x) = \dfrac{\cos(x^2)}{3-x}$ $\quad \left(a=0;\ x=\tfrac{1}{2}\right)$

So $f^{(3)}(c) = \dfrac{\cos(c^2)}{3-c}$ $\quad 0<c<\tfrac{1}{2}$

Can say: $\cos(c^2)\le 1$, $\quad \dfrac{1}{3-c} \le \dfrac{1}{2\frac{1}{2}}$ $\qquad 2\tfrac{1}{2} < 3-c < 3$

$$\Downarrow$$

$$\frac{1}{3} < \frac{1}{3-c} < \frac{1}{2\frac{1}{2}}$$

So $\dfrac{\cos(c^2)}{3-c} \le 1\cdot\dfrac{1}{2\frac{1}{2}} = \dfrac{2}{5}$

So $\left|R_2\left(\tfrac{1}{2}\right)\right| \le \dfrac{1}{3!}\cdot\dfrac{2}{5}\cdot\left(\tfrac{1}{2}\right)^3$ $\qquad R_2(x) = \dfrac{1}{3!} f^{(3)}(c)\,(x-a)^3$

Also true: $\cos(c^2)\le 2$, $\quad \dfrac{1}{3-c} \le \dfrac{1}{2}$ $\qquad$ So $f^{(3)}(c) \le 2\cdot\tfrac{1}{2} = 1$

Example: $f^{(3)}(c) = e^c(c^2-c)$, $\quad 0<c<1$.

Need estimate for $|f^{(3)}(c)|$.

Note: $e^c < e^1$ $\quad |$ Because $0<c<1$, $c^2<c$,

$|c^2-c| = c-c^2 < 1-0 = 1$

So $|f^{(3)}(c)| < e^1\cdot 1 = e$

Or: $c-c^2 = \frac{1}{4} - \left(\frac{1}{4} - c + c^2\right) = \frac{1}{4} - \left(c-\frac{1}{2}\right)^2 \le \frac{1}{4}$

So $|e^c(c^2-c)| \le \dfrac{e}{4}$ $\quad \forall \quad 0<c<1$ $\qquad \boxed{\tfrac{1}{2}<1}$

Q: Limits

Evaluate $\lim_{x\to -1} \frac{\sqrt{x^2+8}-3}{x+1}$

This has form $\frac{0}{0}$ $\quad \sqrt{(-1)^2+8}-3=0$, $\quad -1+1=0$

(1) Algebra: $\frac{\sqrt{x^2+8}-3}{x+1} = \frac{x^2+8-9}{x+1}\cdot\frac{1}{\sqrt{x^2+8}+3}$

$$= \frac{x^2-1}{x+1}\cdot\frac{1}{\sqrt{x^2+8}+3} = \frac{x-1}{\sqrt{x^2+8}+3} \xrightarrow[x\to -1]{} \frac{-2}{3+3} = -\frac{1}{3}.$$

(2) Calculus: let $f(x)=\sqrt{x^2+8}$, then $f(-1)=3$

so we have $\lim_{x\to -1}\frac{f(x)-f(-1)}{x-(-1)} = f'(-1) = \left[\frac{2x}{2\sqrt{x^2+8}}\right]_{x=-1} = \frac{-1}{\sqrt{(-1)^2+8}} = -\frac{1}{3}$

(3) L'Hôpital's rule: since $\lim_{x\to -1}\sqrt{x^2+8}-3=0$, $\lim_{x\to -1} x+1=0$

so by l'Hôpital's rule

$$\lim_{x\to -1}\frac{\sqrt{x^2+8}-3}{x+1} = \lim_{x\to -1}\frac{2x/2\sqrt{x^2+8}}{1} = \lim_{x\to -1}\frac{x}{\sqrt{x^2+8}} = -\frac{1}{3}$$

(4) Taylor expansion: $\sqrt{x^2+8} = \sqrt{((x+1)-1)^2+8} = \sqrt{9-2(x+1)+(x+1)^2}$

Now $\sqrt{9+u} \approx 3+\frac{1}{6}u$ $\quad \left(\sqrt{9}=3,\ \frac{1}{2\sqrt{9}}=\frac{1}{6} \Rightarrow \text{linear approx}\right)$

so $\sqrt{9+(-2(x+1)+(x+1)^2)} \approx 3+\frac{1}{6}\left(-2(x+1)+(x+1)^2\right)$ to 1st order

80 $$\frac{\sqrt{x^3+8}-3}{x+1} \approx \frac{3+\frac{1}{6}\left(-2(x+1)+(x+1)^2\right)-3}{x+1} \approx -\frac{1}{3}+\frac{1}{6}(x+1)$$

$x \to -1$

$-\frac{1}{3}$

IVT: If f is cts, on $[a,b]$ $f(a) < t < f(b)$

then $f(c)=t$ for some $c \in (a,b)$

Use it: to show f actually achieves a value (eg. to show equation has a solution)

$(\tan c = c+1 \iff \tan c - (c+1) = 0)$

MVT: If f is diff on $[a,b]$ then have $a < c < b$ s.t. $$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Use it: to understand $f(b)-f(a)$ from information about f'.

E.g.: If $f' > 0$ on (a,b) then $f(b) > f(a)$

If $f' < 100$ then $f(b) < f(a) + 100(b-a)$

Midterms show that $\tan c = c+1$ for some c

let $f(x) = \tan x - (x+1)$

f not cts everywhere!

$$\lim_{x\to -\frac{\pi}{2}^+} f(x) = \left(\lim_{x\to -\frac{\pi}{2}^+} \tan x\right) - \lim_{x\to -\frac{\pi}{2}^+} (x+1) = -\infty$$

$$\lim_{x\to \frac{\pi}{2}^-} f(x) = \left(\lim_{x\to \frac{\pi}{2}^-} \tan x\right) - \lim_{x\to +\frac{\pi}{2}^-} x+1 = +\infty$$

So choose $a > -\frac{\pi}{2}^-$ but close to $-\frac{\pi}{2}$ st $f(a) < 0$

Similarly choose $b < \frac{\pi}{2}$, but close to $\frac{\pi}{2}$, s.t. $f(b) > 0$

Because $-\frac{\pi}{2} < a < b < \frac{\pi}{2}$, f is cts on $[a,b]$ (defined by formula there)

By IVT there is c, $a < c < b$ s.t. $f(c) = 0$,

that is $\tan c - (c+1) = 0$.

Therefore also $\tan c = c+1$.

Example show $2x^2 - 3 + \sin x + \cos x = 0$ has exactly two solutions

let $f(x) = 2x^2 - 3 + \sin x + \cos x$.

$\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to -\infty} f(x) = \infty$ so have $a<0$, $b>0$

st $f(a)$, $f(b) > 0$ (or take $a = -10$, $b = 10$)

$f(0) = -3+1 = -2 < 0$, Since f is cts everywhere (defined by formula), by IVT it has a zero in $(a,0)$, and in $(0,b)$

Also, $f''(x) = 4 - \sin x - \cos x \geq 4 - 1 - 1 = 2 > 0$

So f ~~is~~ is concave up, can only meet a line in two points (secant lines are above graph)

so f has exactly two zeroes.

Example (4) find c s.t. $3c = c^2$

~~let~~ let $f(x) = x^2 - 3x$ which is cts everywhere (polynomi

$$f(10) = 100 - 30 = 70 > 0$$

$$f(-10) = +100 + 30 = 130 > 0$$

$$f(1) = 1 - 3 = -2$$

so by IVT ~~so~~ f has a zero on $(-10, 1)$, and a zero on $(1, 10)$

if $f(c) = 0$ then $3c = c^2$.