Math 100, lecture 22, 30/11/2021

Last time: Anti-derivatives

Reverse diff: given f' find f.

Idea: massage expression until they work.

Review

Course has:
(1) computing limits, continuity
(2) definition of derivative + diff rules.
(3) applications of derivative:
related rates, implicit diff,
optimization, linear + non-linear approx,
Taylor expansion, curve sketching, ..
(4) Anti-derivatives.

Q: how do we find the objective function for optimization?

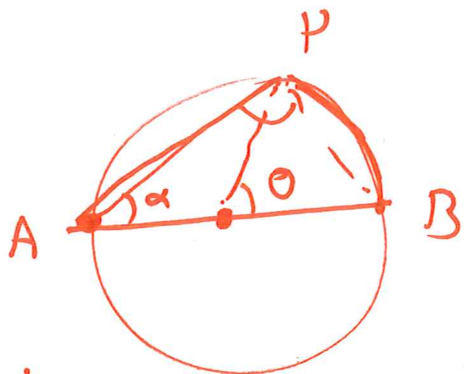
Suggestion: (0) parse question
(1) draw diagram, assign names to quantities
(2) figure out ~~the~~ relations between quantities
find target quantity as a function of other quantities,
use other relations to eliminate all but one variable

Example 6: road + bridge:

Start cost $C = C\left(\begin{matrix}\text{length} \\ \text{of road}\end{matrix}, \begin{matrix}\text{length of} \\ \text{bridge}\end{matrix}\right)$

use geometry to write length of bridge in terms of length of road

<u>webworks</u>



~~prob~~ want to cross from A to B (diameter of circle) can row to P then run around lake.

Objective is $\text{time} = \dfrac{\text{rowing distance}}{\text{rowing speed}} + \dfrac{\text{running distance}}{\text{running speed}}$

use α, or θ as parameter, determine distances in terms of the parameter:

row distance of $2r\cos\alpha$

run distance of $r\theta$

<u>facts</u> $\theta = 2\alpha$

Q: What may we use when computing limits?

A: everything:

(1) algebra

(2) definition of derivative

(3) Taylor expansion

(4) l'Hôpital's rule

(5) squeeze thm

(6) continuity

Examples $\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\sin x - \sin 0}{x - 0} = \left[\frac{d}{dx}\sin x\right]_{x=0} = \cos 0 = 1$

or $\lim_{x\to 0} \frac{\sin x}{x} \underset{\text{l'Hôpital}}{=} \lim_{x\to 0} \frac{\cos x}{1} = \frac{\cos 0}{1} = 1$

$$\begin{cases} \lim_{x\to 0} \sin x = \sin 0 = 0 \\ \lim_{x\to 0} x = 0 \end{cases}$$

[ok to know that $\lim_{x\to 0} \frac{\sin x}{x} = 1$]

eg. $\lim_{t\to -3} \frac{(2t+6)}{\sqrt{t+4} - 1}$

(1) algebra: $\frac{2t+6}{\sqrt{t+4}-1} = \frac{2t+6}{t+4-1}\cdot\left(\sqrt{t+4}+1\right)$

$= 2\left(\sqrt{t+4}+1\right) \xrightarrow[t\to -3]{} 4$

(2) $\lim_{t\to -3} \frac{\sqrt{t+4}-1}{2t+6} = \frac{1}{2}\lim_{t\to -3} \frac{\sqrt{t+4} - \sqrt{(-3)+4}}{t-(-3)} = \frac{1}{2}\left[\frac{d}{dt}\sqrt{t+4}\right]_{t=-3}$

$= \frac{1}{2}\left[\frac{1}{2\sqrt{t+4}}\right]_{t=-3} = \frac{1}{4} \Rightarrow \lim_{t\to -3} \frac{2t+6}{\sqrt{t+4}-1} = \frac{1}{1/4} = 4.$

(3) $\sqrt{t+4} = \sqrt{1+(t+3)} \approx 1+\frac{1}{2}(t+3)$ to 1^{st} order

$$\left[\frac{d}{dx}\sqrt{1+x}\right]_{x=0} = \frac{1}{2}$$

So $\dfrac{2t+6}{\sqrt{t+4}-1} \approx \dfrac{2t+6}{(1+\frac{1}{2}(t+3)-1)} = 4$ to 0^{th} order

or. $\dfrac{2t+6}{\sqrt{t+4}-1} = \dfrac{2t+6}{1+\frac{1}{2}(t+3)+R_1(t)-1} = \dfrac{2}{\frac{1}{2}+\frac{R_1(t)}{t+3}} \to \dfrac{2}{1/2}$

$$\frac{R_1(t)}{t+3} \xrightarrow[t\to-3]{} 0$$

(4) $\lim\limits_{t\to-3} 2t+6 = 2\cdot(-3)+6 = 0$ so by l'Hôpital's rule

$\lim\limits_{t\to-3} \sqrt{t+4}-1 = \sqrt{1}-1 = 0$

$$\lim_{t\to-3} \frac{2t+6}{\sqrt{t+4}-1} = \lim_{t\to-3} \frac{2}{\frac{1}{2\sqrt{t+4}}} = \lim_{t\to-3} 4\sqrt{t+4} = 4.$$

Morally speaking: numerator, $2t \approx -6$, $2t+6 \approx 2(t+3)$

denominator. $\sqrt{t+4} \approx 1$, $\sqrt{t+4}-1 \approx \frac{1}{2}(t+3)$

squeeze thm

compare two limits

(1) $\lim_{x \to 0^+} x \log x$ — forces $x \to 0$, $\log x \to -\infty$ — who wins?

(2) $\lim_{x \to 0^+} x \sin\left(\frac{1}{x}\right)$ — $x \to 0$, $\sin\left(\frac{1}{x}\right)$ bounded — clear x wins

squeeze thm is use to justify the fact that x wins in (2)

have $-x \le x\sin\left(\frac{1}{x}\right) \le x$ (if $x>0$)

easier to control $x, -x$ than $x\sin\left(\frac{1}{x}\right)$.

For (1) we used l'Hôpital: $x \log x = \frac{\log x}{1/x}$ — $\log x \to -\infty$, $1/x \to \infty$ — justification for l'Hôpital

so $\lim_{x \to 0^+} x \log x = \lim_{x \to 0^+} \frac{1/x}{-1/x^2} = \lim_{x \to 0^+} -x = 0$

Notes $x \log x < 0$ if $0 < x < 1$ write $x = \frac{1}{y}$

lookin at $\lim_{y \to \infty} \frac{1}{y} \log \frac{1}{y} = -\lim_{y \to \infty} \frac{1}{y} \log y$

either use l'Hôpital or prove $\log y < \sqrt{y}$ if y is big

so $0 < \frac{1}{y} \log y < \frac{\sqrt{y}}{y}$ use squeeze thm.

Problem

$\tan c = C = 2$

Show that the equation ~~$\tan x = x + 2$~~ has a solution

or $\tan x = x + 2$

Solution:

Let $f(x) = \tan x - x$ goal: find c s.t. $f(c) = 2$

[or $f(x) = \tan x - x - 2$, look for c s.t. $f(c) = 0$]

We'll need to check continuity but f is discontinuous at $\pm\frac{\pi}{2}$.

Notice: $\tan x \underset{x \to \frac{\pi}{2}^-}{\longrightarrow} \infty$, $\tan x \underset{x \to -\frac{\pi}{2}^+}{\longrightarrow} -\infty$

& $x \to \pm\frac{\pi}{2}$ is bounded So:

(1) f is cts on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ (defined by formula, makes sense there)

(2) $\lim_{x \to -\frac{\pi}{2}^+} f(x) = -\infty$ ($\tan x$ blows up, x bounded)

So there is a close to $-\frac{\pi}{2}$, $a > -\frac{\pi}{2}$, s.t. $f(a) < 2$

(3) $\lim_{x \to \frac{\pi}{2}^-} f(x) = +\infty$ ($\tan x$ blows up, x bounded)

So there is b, close to $\frac{\pi}{2}$, $b < \frac{\pi}{2}$, s.t. $f(b) > 2$.

So: $f(a) < 2 < f(b)$, $-\frac{\pi}{2} < a < b < \frac{\pi}{2}$, $\Rightarrow f$ is cts on $[a,b]$

By IVT there is $a < c < b$ s.t. $f(c) = 2$,

ie $\tan c = c + 2$.