# 8. Exponential and Trig functions (7/10/2021)

Goals.

(1) Exponential functions
(2) Trig functions: the definition; their derviatives

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Last Time.

Differentiation rules: $(af+bg)' = af' + bg'$

$(fg)' = f'g + fg'$, $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

$(fg)' \neq f'g'$

(can deduce these from the linear approximation)

Tangent line: line $y = f'(a)(x-a) + f(a)$

determined by/determines (1) y-value at $a = f(a)$

(2) slope at $a = f'(a)$

(1) gives f, $a$ can find line by calculating $f(a)$, $f'(a)$

(2) The line $y = \pi x - e$ is tangent to the graph of $y = f(x)$ at $x = -1$. What are $f(-1)$, $f'(-1)$?

* $f'(-1) = \text{slope} = \pi$.
* $f(-1) = y(-1) = -\pi - e$ (graphs of f, line meet at $x = -1$)

<u>Idea:</u> if we don't know the value of a quantity, give it a <u>name</u>, calculate with the unknown value.

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## <u>Exponential function</u>

Has the form $x \mapsto q^x$, where $q>0$ is called the base.

<u>Facts:</u> $q^{x+y} = q^x \cdot q^y$, $q^{-x} = \frac{1}{q^x}$

$(q^x)^y = q^{xy}$, $(qr)^x = q^x \cdot r^x$.

<u>warning:</u> $q^{(x^y)} \neq (q^x)^y$, $q^{x^y} = q^{(x^y)}$

<u>inverse:</u> <u>logarithm</u>: if $y = q^x$ then

$$x = \log_q y.$$

<u>laws:</u> $\log_q(xy) = \log_q x + \log_q y$, $\log_q(x^y) = y \log_q x$

$\log_r x = \frac{\log_q x}{\log_q r}$. | $x = r^{\log_r x}$

take $\log_q x = \log_r x \cdot \log_q r$

$\log_{10}(10e^5) = 1 + 5\log_{10}(e)$

Math 100 – WORKSHEET 8
EXPONENTIAL AND TRIG FUNCTIONS

# 1. Exponentials

(1) Simplify

(a) $(e^5)^3$, $(2^{1/3})^{12}$, $7^{3-5}$.

$(e^5)^3 = e^{5\cdot 3} = e^{15}$, $(2^{1/3})^{12} = 2^{\frac{1}{3}\cdot 12} = 2^4 = 16$, $\frac{7^3}{7^5} = 7^{3-5} = 7^{-2}$

(b) $\log(10e^5)$, $\log(3^7)$.

def of log is that $\log(e^5) = 5$

$\log(10e^5) = \log 10 + \log(e^5) = \log 10 + 5$

$\log(3^7) = 7\log 3$

(2) Differentiate:

$e \approx 2.71828$

$10 \approx e^{2.3}$

$\log 10 \approx 2.3$

(a) $10^x$

$\frac{d}{dx} 10^x = (\log 10)\cdot 10^x$

(b) $\frac{5\cdot 10^x + x^2}{3^x+1}$ quotient rule

$$\left(\frac{5\cdot 10^x + x^2}{3^x+1}\right)' = \frac{(5\cdot 10^x + x^2)'\cdot(3^x+1) - (5\cdot 10^x + x^2)\cdot(3^x+1)'}{(3^x+1)^2} =$$

$$= \frac{(5\cdot\log 10\cdot 10^x + 2x)(3^x+1) - (5\cdot 10^x + x^2)\cdot\log 3\cdot 3^x}{(3^x+1)^2}$$

*Date*: 7/10/2021, Worksheet by Lior Silberman. 

We will use $\log = \log_e$ (other fields use $\log = \log_{10}$ or $\log = \log_2$). Why?

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**What is $\frac{d}{dx} q^x$ ?**

By definition, it is $\lim\limits_{h \to 0} \frac{q^{x+h} - q^x}{h} =$

$$= \lim_{h \to 0} \frac{q^x q^h - q^x}{h} = \lim_{h \to 0} q^x \frac{q^h - 1}{h} = q^x \lim_{h \to 0} \frac{q^h - 1}{h}$$

$\Rightarrow$ write $L(q) = \left[(q^x)'\right]_{x=0}$

Then $\boxed{\frac{d}{dx} q^x = L(q) \cdot q^x.}$

checks $L(qr) = L(q) + L(r)$

$L(q^t) = t L(q)$

$$L(qr) = \left[\frac{d}{dx} (qr)^x\right]_{x=0} = \left[\frac{d}{dx} (q^x r^x)\right]_{x=0} \cdots$$

Fact: There is a number $e$ s.t. $L(e) = 1$

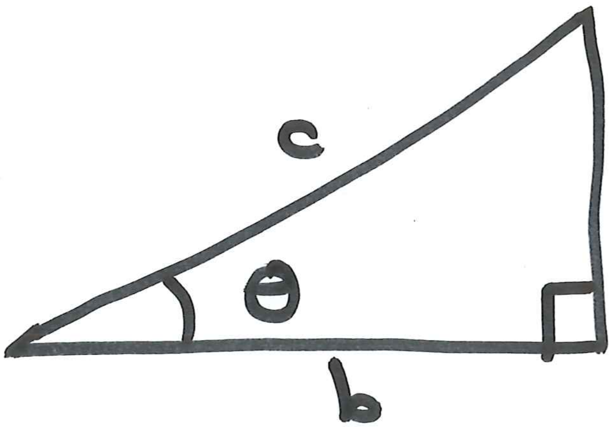
Call $e$ "natural base of the logarithm".

natural because $(e^x)' = e^x$. $\Rightarrow L(q) = \log_e q = \log q$

$(q^x)' = (\log q) \cdot q^x$

# Trig functions

in a right-angled triangle

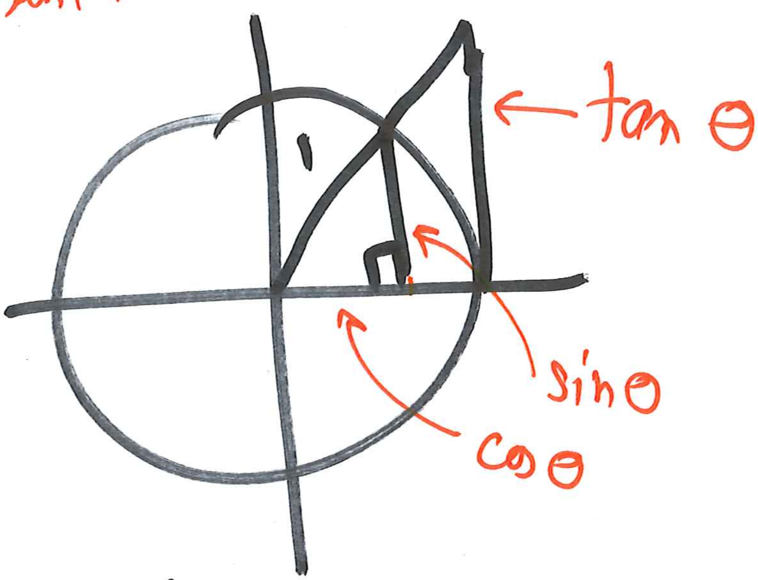



$$\frac{b}{c} = \cos\theta$$

$$\frac{a}{c} = \sin\theta$$

$$\frac{a}{b} = \frac{\sin\theta}{\cos\theta} = \tan\theta$$

"Cosine"
"sine"
"tangent".




**Facts:** the circle has $2\pi$ radians

measure angle by arc length around circle.

<u>never in degrees.</u>

Facts: (1) period of sin / cos is $2\pi$

of tan is $\pi$

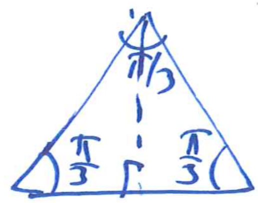
(2) graph

(3) standard values: at $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \ldots$

$\cos\frac{\pi}{6}$

e.g. $= \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$\sin\frac{\pi}{6} = \cos\frac{\pi}{3} = \frac{1}{2}$




$\sin\frac{\pi}{3} = \sin 60^\circ = \frac{\sqrt{3}}{2}$

$\sin\left(\frac{\pi}{2} - \alpha\right) = \cos\alpha$ ..

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Facts: if $\theta$ is measured in radians,

$\boxed{\frac{d}{d\theta}\sin\theta = \cos\theta}$, $\boxed{\frac{d}{d\theta}\cos\theta = -\sin\theta}$

## 2. Trigonometric functions

(3) (Special values) What is $\sin\frac{\pi}{3}$? What is $\cos\frac{5\pi}{2}$?

need: $\frac{\sin h}{h}$, $\frac{f(h)-f(0)}{h}$

(4) Derivatives of trig functions

(a) Interpret $\lim_{h\to 0}\frac{\sin h}{h}$ as a derivative and find its value.

let $f(x) = \sin x$, at $a=0$ $\quad f'(0) = \lim_{h\to 0}\frac{f(h)-f(0)}{h} = \lim_{h\to 0}\frac{\sin h}{h}$

so $\lim_{h\to 0}\frac{\sin h}{h} = f'(0) = \cos 0 = 1$

(b) Differentiate $\tan\theta = \frac{\sin\theta}{\cos\theta}$.

$$\frac{d}{d\theta}\tan\theta = \frac{(\sin\theta)'\cdot\cos\theta - \sin\theta\,(\cos\theta)'}{\cos^2\theta} = \frac{\cos^2\theta+\sin^2\theta}{\cos^2\theta}$$

quotient rule

$$= \frac{1}{\cos^2\theta} = 1+\tan^2\theta$$

both are true

Q: have limit $\lim_{h \to 0} \frac{\sin h}{h}$

want it to have the form $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

need to choose $f$, $a$.

See "$\sin h$" try: $f(x) = \sin x$, $a = 0$

check: indeed $\sin 0 = 0$ so

$$\frac{f(h+0) - f(0)}{h} = \frac{\sin h - \sin 0}{h} = \frac{\sin h}{h}$$

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Appendix to CLP has list of high school facts

## 3. Functions in Chains

(6) Write each function as a composition $f$ is a composite function

(a) $e^{3x}$

$f(x) = e^{3x}$ $\quad f(x) = e^{(3x)} = g(h(x))$

where $g(u) = e^u$, $h(x) = 3x$

(b) $\sqrt{2x+1}$

(c) (Final, 2015) $\sin(x^2)$

(d) $(7x + \cos x)^n$.