MATH 100, lecture 24, 7/12/2021

Review 3

Q: NLC

A turkey is taken out of the oven when its temperature is 75°C, placed in a 25°C room. It is cooling at the rate of 1°/min when its temperature is 50°C. When will it reach 40°C?

Let $y(t) = T_{\text{turkey}}(t) - 25°C$, t measured in minutes since the turkey was taken out of the oven

We know: (1) $y(0) = 75 - 25 = 50°C$

(2) when $y(t) = 25°C$ also have $y'(t) = -1°/\text{min}$

Question: when is $y(t) = 15°C$

NLC says: $y(t)$ decays exponentially, so $y(t) = 50 \cdot e^{-kt}$ for some k

Let s be the time when $y(s) = 25°C$ so $y'(s) = -1$

~~then~~ so $50 \cdot e^{-ks} = 25$; $-50ke^{-ks} = -1$

$y'(t) = -50ke^{-kt}$

If $50e^{-k3} = 25$ then $k = \frac{1}{50e^{-k3}} = \frac{1}{25}$

so $y(t) = 50 \cdot e^{-t/25}$

Finally, $y(t) = 15$ when $15 = 50 \cdot e^{-t/25}$

so $t = 25 \log\left(\frac{50}{15}\right) = 25 \log\left(\frac{10}{3}\right)$

(or $t = -25 \log \frac{15}{50}$)

Can state NLC: $\frac{d}{dt} T(t) = -k \cdot (T(t) - T_{env})$

2: Taylor remainder

Let $f(x) = \log x$, $T_n(x)$ the nth degree Taylor expansion about $x = 1$. For which n is $T_n(1.1)$ an overestimate/underestimate of $f(1.1) = \log(1.1)$

Solution: $f'(x) = \frac{1}{x}$, $f^{(2)}(x) = -\frac{1}{x^2}$, $f^{(3)}(x) = \frac{1 \cdot 2}{x^3}$,

$f^{(4)}(x) = -\frac{1 \cdot 2 \cdot 3}{x^4}$, $f^{(5)}(x) = +\frac{1 \cdot 2 \cdot 3 \cdot 4}{x^5}$, ...

We see, $f^{(n+1)}(x) = \frac{1 \cdot 2 \cdots n}{x^{n+1}} \cdot \begin{cases} 1 & n+1 \text{ odd} \\ -1 & n+1 \text{ even} \end{cases}$

By Lagrange form of the remainder,

$$R_n(1.1) = \frac{f^{(n+1)}(c)}{(n+1)!}(1.1-1)^{n+1} \quad \text{when} \quad 1 < c < 1.1$$

So $R_n(1.1) = \frac{n!}{(n+1)!} \cdot \frac{1}{c^{n+1}} (0.1)^{n+1} \cdot \begin{cases} 1 & n \text{ even} \\ -1 & n \text{ odd} \end{cases}$, $1 < c < 1.1$

positive positive positive

So $R_n(1,1) > 0$ if n even

$R_n(1.1) < 0$ if n is odd

So $T_n(1,1)$ is an underestimate if n is even

an overestimate if n is odd

Q1 (Algebra)

Let $g(y)$ be the ~~answer~~ function Inverse to $f(x) = e^x + x^e$

find $g(2e^e)$.

$g(2e^e) = x$ if $f(x) = 2e^e$ so we need x s.t. $e^x + x^e = 2e^e$.

Well, $e^e + e^e = 2e^e$ so $g(2e^e) = e$.

Q: Derivatives:

differentiate

$$f(x) = \left[\arcsin\left(\frac{x}{\sin x}\right)\right]^{\log(\sin e^x)}$$

Warning: $\frac{x}{\sin x} > 1$ iff $x \neq 0$ so this expression has domain $\{0\}$, no derivative (error in the original) (but using $\frac{\sin x}{x}$ would have been fine)

Then $\log f = \log(\sin e^x) \cdot \log \arcsin\left(\frac{x}{\sin x}\right)$

So $\frac{f'}{f} = (\log f)' = (\log(\sin e^x))' \cdot \log \arcsin\left(\frac{x}{\sin x}\right)$

pdt rule

$+ \log(\sin e^x) \cdot \left(\log \arcsin\left(\frac{x}{\sin x}\right)\right)'$

chain rule

$$= \left(\frac{1}{\sin(e^x)}\left(\sin(e^x)\right)'\right) \cdot \log \arcsin\left(\frac{x}{\sin x}\right) + \log(\sin e^x) \cdot \frac{1}{\arcsin \frac{x}{\sin x}} \cdot \left(\arcsin \frac{x}{\sin x}\right)'$$

$$= \frac{\cos(e^x)}{\sin(e^x)} (e^x)' \cdot \log \arcsin\left(\frac{x}{\sin x}\right) + \log(\sin e^x) \cdot \frac{1}{\arcsin \frac{x}{\sin x}} \cdot \frac{1}{\sqrt{1-\left(\frac{x}{\sin x}\right)^2}} \left(\frac{x}{\sin x}\right)'$$

$$= \frac{\cos(e^x)}{\sin(e^x)} e^x \cdot \log \arcsin\left(\frac{x}{\sin x}\right) + \log(\sin e^x) \cdot \frac{1}{\arcsin \frac{x}{\sin x}} \cdot \frac{1}{\sqrt{1-\left(\frac{x}{\sin x}\right)^2}} \cdot \left(\frac{1}{\sin x} - \frac{x \cos x}{\sin^2 x}\right)$$

quotient rule

So

$$f'(x) = \left[\arcsin\left(\frac{x}{\sin x}\right)\right]^{\log(\sin e^x)} \cdot \left[\frac{\cos(e^x)}{\sin(e^x)} e^x \log \arcsin\left(\frac{x}{\sin x}\right) + \frac{\log(\sin(e^x))}{\arcsin \frac{x}{\sin x}} \cdot \frac{1}{\sqrt{1-\left(\frac{x}{\sin x}\right)^2}} \left(\frac{1}{\sin x} - \frac{x \cos x}{\sin^2 x}\right)\right]$$

Q5 IVT

Call two points on Earth antipodal if they are on opposite sides w.r.t centre, show that there are two antipodal points with the same temperature

Solution:

Let $T(\theta)$ be the temperature at the point of longitude θ along the equator.

Let $f(\theta) = T(\theta) - T(\theta + \pi)$ be the temperature difference between the antipodal points θ, $\theta + \pi$

(Assume the $T(\theta)$ is cts, so $f(\theta)$ is cts as well)

Note: $f(\theta + \pi) = T(\theta + \pi) - T(\theta) = -f(\theta)$

Take any point θ_0, if $f(\theta_0) = 0$ we found two antipodal points with same temperature. If not, $f(\theta_0 + \pi) = -f(\theta_0)$ so the have opposite signs

By IVT there is $\theta_0 < \theta < \theta_0 + \pi$ s.t. $f(\theta) = 0$.

Q: inverse fcns

Let $f(x) = e^{x} - e^{-x}$

(1) $(f'(x))^2 = 4 + (f(x))^2$

(2) Show f has an inverse fcn

(3) let $g(y)$ be the inverse fcn, find g' in terms of x, y

(4) find a formula for $g'(y)$ in terms of y.

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① $f'(x) = e^{x} + e^{-x}$ so $f'(x)^2 = e^{2x} + 2 + e^{-2x}$

also $f(x)^2 = e^{2x} - 2 + e^{-2x}$

so $f'(x)^2 = 4 + f(x)^2$.

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② Note that $f'(x) = e^{x} + e^{-x} > 0$ for all x, so f is strictly increasing and takes every value once.

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③ We know: $g'(y) = \dfrac{1}{f'(x)} = \dfrac{1}{e^{x} + e^{-x}}$ by inverse fcn rule

$(y = e^{x} - e^{-y})$

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④ $g'(y) = \dfrac{1}{\sqrt{4 + f(x)^2}} = \dfrac{1}{\sqrt{4 + y^2}}$ since $f'(x) > 0$ for all x

formula from part ①