20. L'Hopital's rule (23/11/2021)

Goals.

(1) Compute limits using l'Hopital's rule
(2) Connect to Taylor expansion

Last Time. Curve sketching.

Question: can a discontinuity be an inflection point? no

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hypotheses

IF $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ (can also have $a = \pm\infty$, or both limits are ∞ in extended sense)

and $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists

conclusion

THEN $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$

To use a theorem need to (1) verify the hypotheses
(2) invoke theorem by name

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limits of form $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0\cdot\infty$, etc are called indeterminate form

Why does it work!

If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ define $f(a) = g(a) = 0$

then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \Big/ \frac{g(x)-g(a)}{x-a}$$

Math 100 – WORKSHEET 20
L'HÔPITAL'S RULE

(1) Evaluate $\lim_{x\to 1} \frac{\log x}{x-1}$.

$\lim_{x\to 1} \log x = \log 1 = 0$; $\lim_{x\to 1} x-1 = 0$

By l'Hôpital's rule, $\lim_{x\to 1} \frac{\log x}{x-1} = \lim_{x\to 1} \frac{1/x}{1} = 1.$

But: $\lim_{x\to 1} \frac{\log x}{x-1} = \lim_{x\to 1} \frac{\log x - \log 1}{x-1} = \left[\frac{d}{dx} \log x\right]_{x=1} = \left[\frac{1}{x}\right]_{x=1} = 1$

(2) (Final, 2014) Evaluate $\lim_{x\to 0} \frac{\cos x - e^{x^2}}{x^2}$.

$\lim_{x\to 0} (\cos x - e^{x^2}) = \cos 0 - e^0 = 0$; $\lim_{x\to 0} x^2 = 0$

By l'Hôpital's rule, $\lim_{x\to 0} \frac{\cos x - e^{x^2}}{x^2} = \lim_{x\to 0} \frac{-\sin x - 2xe^{x^2}}{2x}$ (if limit exists)

$\lim_{x\to 0} (-\sin x - 2xe^{x^2}) = -\sin 0 - 0 = 0$; $\lim_{x\to 0} 2x = 0$

By l'Hôpital's rule again, $\lim_{x\to 0} \frac{\cos x - e^{x^2}}{x^2} = \lim_{x\to 0} \frac{-\sin x - 2xe^{x^2}}{2x} =$

$= \lim_{x\to 0} \frac{-\cos x - 2e^{x^2} - 4x^2e^{x^2}}{2} = \frac{-1-2-0}{2} = -3/2$

Date: 23/11/2021, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

Or: $\lim_{x\to 0} \frac{-\sin x - 2xe^{x^2}}{2x} = \lim_{x\to 0} \left[-\frac{1}{2}\frac{\sin x}{x} - e^{x^2}\right] = -\frac{1}{2} - 1 = -3/2$

$\frac{\sin x}{x} \xrightarrow[x\to 0]{} 1$

(3) Do (2) using a 2nd-order Taylor expansion.

$\cos 0 = 1$; $(\cos')(0) = 0$; $(\cos'')(0) = -\cos 0 = -1$

So to 2^{nd} order $\cos x \approx 1 - \frac{1}{2}x^2$

$[e^u]_{u=0} = 1$ same for derivatives so $e^u \approx 1 + u + \frac{1}{2}u^2$ to 2^{nd} order

So $e^{x^2} \approx 1 + x^2 + \frac{\cancel{x^4}}{2} \approx 1 + x^2$ to 2^{nd} order

So $\cos x - e^{x^2} = (1 - \frac{1}{2}x^2) - (1 + x^2) + R_2(x)$

R_2 is of size x^3

So $\frac{\cos x - e^{x^2}}{x^2} = -\frac{3/2\, x^2}{x^2} + \frac{R_2(x)}{x^2} = -\frac{3}{2} + \frac{R_2}{x^2} \xrightarrow[x \to 0]{} -3/2$

(4) (Final, 2015) Evaluate $\lim_{x\to 0} \frac{\log(1+x) - \sin x}{x^2}$.

$\lim_{x\to 0} \log(1+x) - \sin x = \log 1 - \sin 0 = 0$; $\lim_{x\to 0} x^2 = 0$

$\lim_{x\to 0} \frac{1}{1+x} - \cos x = \frac{1}{1} - \cos 0 = 0$; $\lim_{x\to 0} 2x = 0$

So by l'Hôpital's rule twice we have

$$\lim_{x\to 0} \frac{\log(1+x) - \sin x}{x^2} = \lim_{x\to 0} \frac{\frac{1}{1+x} - \cos x}{2x} = \lim_{x\to 0} \frac{-\frac{1}{(1+x)^2} + \sin x}{2} = -\frac{1}{2}$$

$\log(1+x) \approx x - \frac{x^2}{2}$, $\sin x \approx x - \frac{x^3}{6} \approx x$ to 2^{nd} order

$R_2(x)$ is of size x^3

So $\frac{\log(1+x) - \sin x}{x^2} = \frac{x - \frac{x^2}{2} - x + R_2(x)}{x^2} = -\frac{1}{2} + \frac{R_2(x)}{x^2} \xrightarrow[x\to 0]{} -\frac{1}{2}$

suppose f,g continuously diff near 2.

(5) Given that $f(2) = 5$, $g(2) = 3$, $f'(2) = 7$ and $g'(2) = 4$ find $\lim_{x\to 3} \frac{f(2x-4)-g(x-1)-2}{g(x^2-7)-3}$.

$\lim_{x\to 3} f(2x-4) - g(x-1) - 2 = f(2) - g(2) - 2 = 5-3-2 = 0$

$\lim_{x\to 3} g(x^2-7) - 3 = g(2) - 3 = 0$

f,g cts at 2 since they are diff there

By l'Hôpital's rule,

$$\lim_{x\to 3} \frac{f(2x-4) - g(x-1) - 2}{g(x^2-7) - 3} = \lim_{x\to 3} \frac{f'(2x-4)\cdot 2 - g'(x-1)}{g'(x^2-7)\cdot 2x}$$

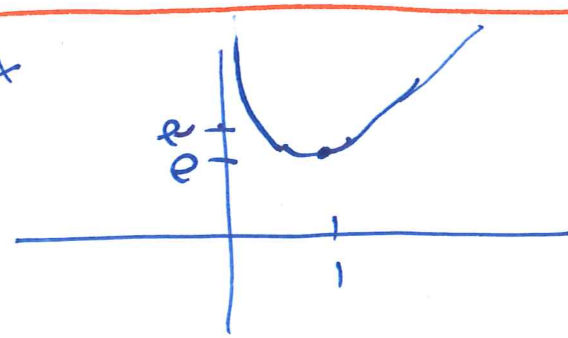
$$= \frac{2f'(2) - g'(2)}{g'(2)\cdot 2\cdot 3} = 5/12$$

(6) Evaluate $\lim_{x\to 0^+} \frac{e^x}{x}$.

$\lim_{x\to 0^+} e^x = e^0 = 1 \neq 0$; $\lim_{x\to 0^+} x = 0$ so $\frac{e^x}{x}$ blows up at 0.

Since $\frac{e^x}{x} > 0$ for $x>0$, $\lim_{x\to 0^+} \frac{e^x}{x} = \infty$

$$\left(\frac{e^x}{x}\right)' = \frac{xe^x - e^x}{x^2} = \frac{(x-1)e^x}{x^2}$$



$$\lim_{x\to\infty} \frac{e^x}{x} \underset{\text{l'Hôpital}}{=} \lim_{x\to\infty} \frac{e^x}{1} = \infty$$

$\lim_{x\to\infty} e^x = \infty$

$\lim_{x\to\infty} x = \infty$

"$\infty \cdot 0$"

$e^x \xrightarrow[x\to\infty]{} \infty$

(7) Evaluate $\lim_{x\to\infty} x^2 e^{-x}$.

$$\lim_{x\to\infty} x^2 e^{-x} = \lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{2}{e^x} = 0$$

l'Hôpital's rule: $\lim_{x\to\infty} x^2 = \infty$, $\lim_{x\to\infty} e^x = \infty$

l'Hôpital: $\lim_{x\to\infty} 2x = \infty$, $\lim_{x\to\infty} e^x = \infty$

Or: $\frac{x^2}{e^x} \xrightarrow[x\to\infty]{} 0$ since e^x dominates the polynomial x^2.

"$0 \cdot \infty$"

(8) Evaluate $\lim_{x\to 0^+} x \log x$.

$$\lim_{x\to 0^+} x \log x = \lim_{x\to 0^+} \frac{\log x}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} -x = 0$$

l'Hôpital: $\lim_{x\to 0^+} \log x = -\infty$, $\lim_{x\to 0^+} 1/x = \infty$

$\boxed{\frac{x^2 \log x}{x} \xrightarrow{\text{l'H}} \frac{2x\log x + x}{1} \quad \text{no progress}}$

$$\lim_{x\to 0^+} \frac{x}{1/\log x} = \lim_{x\to 0^+} \frac{1}{-\frac{1}{\log^2 x}\cdot\frac{1}{x}} = -\lim_{x\to 0^+} x \log^2 x$$

no progress

l'Hôpital: $\lim_{x\to 0^+} x = 0$, $\lim_{x\to 0^+} \frac{1}{\log x} = 0$

Conclusion: $\lim_{x\to 0^+} x^x = \lim_{x\to 0} e^{\overset{\log(x^x)}{\overbrace{x\log x}}} = e^0 = 1$

(9) Evaluate $\lim_{x\to 0}(2x+1)^{1/\sin x}$.

$\log\left((2x+1)^{1/\sin x}\right) = \frac{1}{\sin x}\log(2x+1)$

$\lim_{x\to 0} \frac{\log(2x+1)}{\sin x} = \lim_{x\to 0} \frac{2/(2x+1)}{\cos x} = \frac{2}{2\cdot 0+1}\cdot\frac{1}{\cos 0} = 2$

l'Hôpital

$\lim_{x\to 0} \log(2x+1) = \log 1 = 0$

$\lim_{x\to\infty} \sin x = 0$

e^y is cts

So $\lim_{x\to 0}(2x+1)^{1/\sin x} = \lim_{x\to 0} e^{\log(2x+1)^{1/\sin x}} = e^{\lim_{x\to 0}\log(2x+1)^{1/\sin x}} = e^2$

don't forget to exponentiate back

(10) Evaluate $\lim_{x\to\infty} x^n e^{-x}$.

Alternative:

Let $f(x) = (2x+1)^{1/\sin x}$, then $\log f(x) = \frac{\log(2x+1)}{\sin x} \xrightarrow[x\to 0]{} 2$

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(11) Suppose $a > 0$. Evaluate $\lim_{x\to\infty} x^{-a}\log x$.