18. The shape of the graph (16/11/2021)

Goals.

(1) Midterm review
(2) Implications of MVT for the shape of the graph:
  (a) Increasing and decreasing functions
  (b) Concave and convex functions

~~Last Time~~.

Midterm conclusions: (1) algebra is a major source of errors
(2) checking your work (+ sanity checks)

Last time: optimization.

find singular + critical pts (know these contain all local extrema)

Today: How ~~do~~ do we tell if f has local max at x_0?
(more generally, what does the graph look like?)

Given an expression for $f(x)$, can:

(0) tell where $f(x)<0$, $f(x)>0$, $f(x_0)=0$, vertical, horizontal asymptotes

(1) tell where $f'(x)>0$, $f'(x)<0$, $f'(x_0)=0$, or DNE $\implies$ increase/decrease in f.

(2) tell where $f''(x)>0$, $f''(x)<0$, $f''(x_0)=0$.

Midterm review

$\lim_{x\to-\infty}\sqrt{4x^2-3x}+2x+1 = \lim_{x\to\infty}\sqrt{4x^2-3x}-2x+1 = \lim_{x\to\infty} x\left(\sqrt{4-3/x}-2+\frac{1}{x}\right)$ +

1. Evaluate the following limits:

can't take limit of one part $\to = \lim_{x\to\infty} x\cdot 0 = 0$

(a) $\lim_{x\to-\infty}\sqrt{4x^2-3x}+2x+1$

$$\lim_{x\to-\infty}\sqrt{4x^2-3x}+2x+1 = \left(\sqrt{4x^2-3x}+2x+1\right)\cdot\frac{\sqrt{4x^2-3x}-2x-1}{\sqrt{4x^2-3x}-2x-1} = \frac{4x^2-3x-\cancel{2x^2-1}\ {}^{-(2x+1)^2}}{\sqrt{4x^2-3x}-2x+1} =$$

$$= \frac{2x^2-3x-1}{2x\sqrt{1-\frac{3}{4x}}-2x+1} \overset{\lim_{x\to-\infty}}{=} \lim_{x\to-\infty}\frac{x^2}{x}\cdot\frac{2-\frac{3}{x}-1/x^2}{2\sqrt{1-\frac{3}{4x}}-2+\frac{1}{x}} = \frac{2-0-0}{2\sqrt{1-0}-2} = \frac{2}{2-2} = \frac{2}{0}\ \text{DNE}.$$

$\sqrt{x^2}=-x$ if $x<0$; $-2x$; -1

shouldn't disappear

(b) $\lim_{t\to-3}\frac{2t+6}{\sqrt{t+4}-1}$

$$\lim_{t\to-3}\frac{2t+6}{\sqrt{t+4}-1} = \lim_{t\to-3}\frac{2t+6}{\sqrt{t+4}-1}\cdot\frac{\sqrt{t+4}+1}{\sqrt{t+4}+1} = \lim_{t\to-3}\frac{(2t+6)(\sqrt{t+4}+1)}{t+4-1}\ \begin{matrix}\xrightarrow[t\to-3^+]{}\ \infty\\ \xrightarrow[t\to-3^-]{}\ -\infty\end{matrix}$$

wrong: numerator vanishes too

or: $\lim_{t\to-3} 2\sqrt{t+4}+1 = 2\cdot 1+1 = 3$ (should be $2\cdot(1+1)=4$)

2. Show that there is a number c such that $\tan(c) = c+1$.

Let $f(x) = \tan(x)-(x+1)$. Then $f(0)=1$, $f(\pi) = 0-\pi+1 = -(\pi-1) < 0$

Thus f has a zero between $0,\pi$. (3) didn't involve IVT (4) no endgame

(1) didn't check continuity; (2) f is not cts on $[0,\pi]$: blows up at $\pi/2$

3. Differentiate

(a) $(3+x)^{\frac{3}{x}}$ $(x>3)$ ↓ $3/x$

$$\log(3+x)^{3/x} = \frac{\log 3}{\log x}\log(3+x) \quad \text{so} \quad \left((3+x)^{3/x}\right)' = -\frac{\log 3}{(\log x)^2}\frac{1}{x}\log(3+x) + \frac{\log 3}{(3+x)\log x}$$

← chain rule

but $f' = f\cdot(\log f)'$ $(\log f)'$

(b) $\sin x\cos(x^2+x)$

$$\frac{d}{dx}\left(\sin x\cos(x^2+x)\right) = -\cos x\sin(x^2+x)(2x+2) \qquad (fg)' \neq f'g'$$

5. A population of algae decays exponentially.

(a) If the population falls by a factor of 3 every 30 days, find the time needed for the population to be divided by 2.

$\log 2/3$

$$N = N_0 e^{-kt} \quad N(30) = \frac{2}{3}N_0 \quad \text{so} \quad \frac{2}{3} = e^{-k30} \quad \text{so} \quad \frac{\log 2}{\log 3} = -k\cdot 30 \quad \text{so} \quad k = -\frac{\log 3}{30\log 2}$$

$\leftarrow 1/3$?

$$\text{so } N(t) = \frac{1}{2}N_0 \text{ when } \frac{1}{2} = e^{-kt} \Rightarrow \frac{1}{2} = e^{\frac{\log 3}{\log 2\cdot 30}t} \Rightarrow \log\frac{1}{2} = \frac{\log 3}{\log 2}\frac{t}{30}$$

(b) If the initial population is 100, what is the population after 10 days?

$$\Rightarrow t = 30\cdot\frac{\log\frac{1}{2}\log 2}{\log 3}.$$

$$N(10) = e^{\frac{\log 3}{\log 2}\cdot\frac{10}{30}} = e^{\frac{\log 3}{3\log 2}}.$$

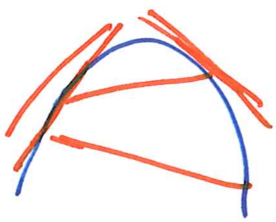
$N(10) > N(0)$??

t can't be negative

$f''(x) > 0 \iff$ "Concave up"

secant lines above
tangent lines below
graph

$f''(x) < 0 \iff$ "cocave down" : mirror image

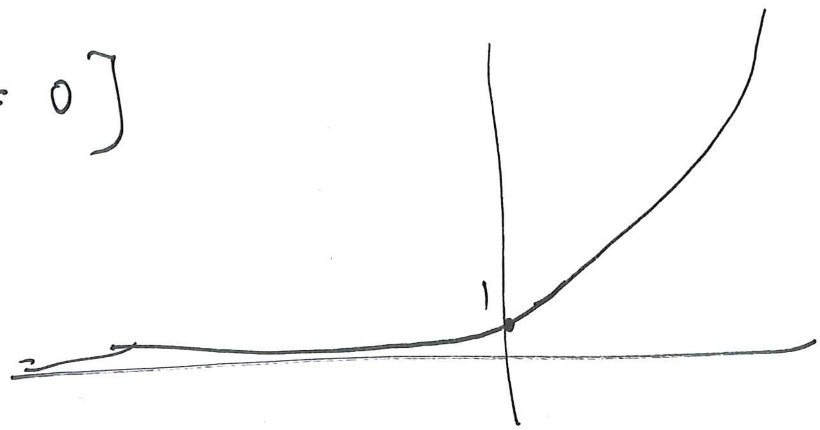


(2) For each of the following functions determine its domain, and where it is increasing or decreasing. Except in part (b) also determine where the function is concave up/down.

(a) $f(x) = e^x$, $f'(x) = e^x$, $f''(x) = e^x$

f, f', f'' always positive, so $f > 0$, increasing, concave up

$\left[\lim_{x \to -\infty} f(x) = 0 \right]$



(b) $f(x) = \frac{x-2}{1+x^2}$

(c) $f(x) = x \log x - 2x$

(d) $\frac{x^2-9}{x^2+3}$. You may use that $f'(x) = \frac{24x}{\left(x^2+3\right)^2}$ and that $f''(x) = 72\frac{1-x^2}{(x^2+3)^3}$.

$f(x) > 0$ on $\{|x|>3\}$ i.e. on $(-\infty,-3)\cup(3,\infty)$, $f(x)<0$ on $(-3,3)$, $f(\pm 3)=0$

f cts everywhere, $\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{1-9/x^2}{1+3/x^2} = 1$

$f'(x) > 0$ if $x>0$, $f'(x)<0$ if $x<0$, $f'(0)=0$: f increasing on $(0,\infty)$, decreasing $(-\infty,0)$

$\Rightarrow$ f has local min at $x=0$

$f''(x)>0$ if $-1<x<1$, $f''(x)<0$ on $(-\infty,-1)\cup(1,\infty)$

$\Rightarrow$ at ± 1 concavity changes: these are inflection points

x	$(-\infty,-3)$	-3	$(-3,-1)$	-1	$(-1,0)$	0	$(0,1)$	1	$(1,3)$	3	$(3,\infty)$
f	+	0	−	−2	−	−3	−	−2	−	0	+
f'	−	−	−	−	−	0	+	+	+	+	+
f''	−	−	−	0	+	+	+	0	−	−	−

$$f(\pm 1) = \frac{1-9}{1+3} = -2$$

$$f(0) = \frac{-9}{3} = -3$$

sketch