15. Taylor Remainder (2/11/2021)

Goals.

(1) Review Taylor expansion
(2) Lagrange remainder for linear approximation
(3) Lagrange remainder: general case

Last Time. Taylor expansion Given f, a:

0) Constant approx: $f(x) \approx f(a)$

1) linear approx: $f(x) \approx f(a) + f'(a)(x-a)$

2) quadratic approx: $f(x) \approx f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2$

3) cubic approx: $f(x) \approx f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2 + \frac{1}{6} f^{(3)}(a)(x-a)^3$

⋮

n'th order correction is $\frac{1}{n!} f^{(n)}(a)(x-a)^n$

$0! = 1$
$1! = 1$

Example: Formula (Einstein): $E(v) = \frac{mc^2}{\sqrt{1-v^2/c^2}}$

energy

m = mass of body, v = velocity
c = speed of light

(aside: only makes sense if $|v| < c$

if $|v| > c$, $1 - \frac{v^2}{c^2} < 0$
if $|v| = c$, $\sqrt{1 - v^2/c^2} = 0$.

$\lim_{v \to c^-} E(v) = \infty$)

Question: What does $\epsilon(v)$ look like for small v?
(Math goal: practice Taylor expansion)

Idea: use $x = v^2/c^2$, $\epsilon(v) = mc^2(1-x)^{-\frac{1}{2}}$

let $f(x) = (1-x)^{-\frac{1}{2}}$, $f'(x) = -\frac{1}{2}(1-x)^{-3/2}(-1)$
$= \frac{1}{2}(1-x)^{-3/2}$

$f''(x) = \frac{1}{2}\cdot\left(-\frac{3}{2}\right)(1-x)^{-5/2}\cdot(-1) = \frac{3}{4}(1-x)^{-5/2}$

$f(0) = 1$, $f'(0) = \frac{1}{2}$, $f''(0) = \frac{3}{4}$.

so $f(x) \simeq 1 + \frac{1}{2}\cdot x + \frac{1}{2!}\cdot\frac{3}{4}\cdot x^2$

so $\epsilon(v) = mc^2 + \frac{1}{2}mc^2\cdot\frac{v^2}{c^2} + \frac{3}{8}mc^2\left(\frac{v^2}{c^2}\right)^2 + \ldots$

$$= mc^2 + \underbrace{\frac{1}{2}mv^2}_{\text{Newtonian kinetic energy}} + \underbrace{\frac{3}{8}m\frac{v^4}{c^2}}_{\text{"first relativistic correction"}} + \ldots$$

Today: Ask "how far off is the approx?"

Write $T_n(x)$ = n'th Taylor polynomial

$R_n(x) = f(x) - T_n(x)$ ← approximation   "error/remainder"

↑ truth

Example: ($n=0$) $f(x) = \underbrace{f(a)}_{T_0(x)} + \underbrace{f'(c)(x-a)}_{R_0(x)}$ c between a, x

Math 100 – WORKSHEET 15
TAYLOR REMAINDER ESTIMATES

1. Review: Taylor expansion

(1) Estimate $(4.1)^{3/2}$ using a linear and a quadratic approximation.

Use $f(x) = x^{3/2}$ about $a=4$. $f'(x) = \frac{3}{2}x^{1/2}$, $f^{(2)}(x) = \frac{3}{4}x^{-1/2}$

$f(4) = 8$, $f'(4) = 3$, $f''(4) = \frac{3}{8}$

To 1^{st} order $f(4.1) \approx 8 + 3(4.1-4) = 8.3$

to 2^{nd} order $f(4.1) \approx 8 + 3 \cdot \frac{1}{10} + \frac{1}{2} \cdot \frac{3}{8} 0.1^2 = 8.3 + \frac{3}{1600}$.

(2) The third-order expansion of $h(x)$ about $x = 2$ is $3 + \frac{1}{2}(x-2) + 2(x-2)^3$. What are $h'(2)$ and $h''(2)$?

$h'(2) = \frac{1}{2}$, $h''(2) = 0$ (no quadratic term)

Date: 2/11/2021, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

2. Error estimate 1

Let $R_1(x) = f(x) - T_1(x)$ be the *remainder*. Then there is c between a and x such that
$$R_1(x) = \frac{f^{(2)}(c)}{2!}(x-a)^2$$

We found $(4.1)^{3/2} \approx 8.3$ to first order.

(4) Estimate the error in the linear approximations to $(4.1)^{3/2}$

By the Lagrange remainder formula,

$$R_1(4.1) = \frac{1}{2!}\left(\frac{3}{4}c^{-1/2}\right)\cdot(4.1-4)^2 = \frac{3}{800}c^{-1/2}, \qquad 4<c<4.1$$

if $4<c<4.1$, $\quad 2<\sqrt{c} \Rightarrow \frac{1}{\sqrt{c}} < \frac{1}{2}$

$$\Rightarrow R_1(4.1) < \frac{3}{800}\cdot\frac{1}{2} = \frac{3}{1600}$$

↑

Estimate (= upper bound) for the error

(also $c < 4.1$ so $R_1(4.1) > \frac{3}{800}\cdot\frac{1}{\sqrt{4.1}}$)

(also $R_1(4.1)>0$, i.e. $f(4.1) > 8.3$
say 8.3 is an underestimate)

Example: Suppose $f^{(4)}(x) = \dfrac{\cos(x^2)}{3-x}$

Question: Estimate the error in a cubic approx to $f(\frac{1}{2})$ about ~~x~~ $a=0$.

By the Lagrange remainder formula

$$R_3(\tfrac{1}{2}) = \frac{1}{4!}\,\frac{\cos(c^2)}{3-c}\cdot\left(\frac{1}{2}\right)^4, \quad 0<c<\frac{1}{2}$$

Here, $\cos(c^2) \le 1$

$c \le \frac{1}{2}$ so $3-c \ge 2\frac{1}{2}$ so $\dfrac{1}{3-c} \le \dfrac{1}{3-\frac{1}{2}}$ $\Big\}$ $\Rightarrow \dfrac{\cos(c^2)}{3-c} \le \dfrac{1}{2.5}$

So $\left|R_3(\tfrac{1}{2})\right| \le \dfrac{1}{24}\cdot\dfrac{1}{2\frac{1}{2}}\cdot\dfrac{1}{16} = \dfrac{1}{960}$

3. Higher order error estimates

> Let $R_n(x) = f(x) - T_n(x)$ be the *remainder*. Then there is c between a and x such that
> $$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

We found $(4.1)^{3/2} \approx 8.301875$ to second order.

(6) Estimate the magnitude of the error in the quadratic approximation to $(4.1)^{3/2}$.

$f^{(3)}(x) = -\frac{3}{8} x^{-3/2}$

By the Lagrange remainder formula

$$R_2(4.1) = \frac{1}{3!}\left(-\frac{3}{8} c^{-3/2}\right) \cdot (4.1-4)^3 = -\frac{1}{16{,}000} c^{-3/2}, \quad 4 < c < 4.1$$

since $c > 4$, $c^{-3/2} < 4^{-3/2} = \frac{1}{8}$

(aside, $R_2(4.1) < 0$ so $(4.1)^{3/2} < 8.301875$)

so $|R_2(4.1)| = \frac{1}{16{,}000} \cdot c^{-3/2} < \frac{1}{16{,}000} \cdot \frac{1}{8} = \frac{1}{128{,}000} < 10^{-5}$